QUANTUM THEORY 2026
ASSIGNMENT 1 (DUE MONDAY 6 APRIL 5PM)

(1) Let $\mathrm{Dom} \subseteq \mathcal{H}$ be a linear subspace of a Hilbert space, and let $A, A^\dagger : \mathrm{Dom} \to \mathrm{Dom}$ be linear operators which are formally adjoint to each other,

$$\langle A^\dagger \varphi | \psi \rangle = \langle \varphi | A\psi \rangle, \qquad \forall\, \varphi, \psi \in \mathrm{Dom}.$$

Show that:

- $A^\dagger A$ and $AA^\dagger$ are formally self-adjoint operators on Dom, i.e.,
$$\langle A^\dagger A\varphi | \psi \rangle = \langle \varphi | A^\dagger A\psi \rangle,$$
$$\langle AA^\dagger \varphi | \psi \rangle = \langle \varphi | AA^\dagger \psi \rangle, \qquad \forall\, \varphi, \psi \in \mathrm{Dom}.$$
- The eigenvalues of $A^\dagger A$ and $AA^\dagger$ are real and non-negative.
- $A^\dagger A$ and $AA^\dagger$ have the same *non-zero* eigenvalues, and that for each $\lambda \neq 0$ eigenvalue, their eigenspaces have the same (possibly infinite) linear dimension.
- $D := \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}$ is formally self-adjoint on $\mathrm{Dom} \oplus \mathrm{Dom}$.
- $\ker(D^2) = \ker(D) = \ker(A) \oplus \ker(A^\dagger)$.
- The point spectrum $\sigma_{\mathrm{p}}(D)$ is symmetric about 0, i.e.,
$$\sigma_{\mathrm{p}}(D) = -\sigma_{\mathrm{p}}(D).$$

(2) Let V be an inner product space, and $W, W' \subset V$ be linear subspaces.

(a) Show that:
- $W \subset W^{\perp\perp}$.
- If $W \subset W'$, then $W'^{\perp} \subset W^{\perp}$.
- $W^{\perp} = W^{\perp\perp\perp}$.

(b) If V is a Hilbert space, show that
- $W^{\perp}$ is a closed subspace.
- $W^{\perp} = \overline{W}^{\perp}$.
- $\overline{W} = W^{\perp\perp}$.

(3) Verify that the functions

$$e_n : x \mapsto \sqrt{2}\cos(\pi n x), \qquad n = 1, 2, \ldots,$$

form an orthonormal set in $L^2(0,1)$. It seems reasonable to define a momentum operator $P = -i\frac{d}{dx} : \mathrm{span}\{e_n : n = 1, 2, \ldots\} \to L^2(0,1)$. Investigate whether or not P is formally self-adjoint on its domain of definition.

(4) Let c_{00} be the vector space of $\mathbb{C}$-valued sequences $s = (s_1, s_2, \ldots)$ with finitely-many nonzero entries. Equip it with the inner product $\langle s | s' \rangle = \sum_{n=1}^{\infty} \overline{s_n} s'_n$. Consider the linear functional

$$\eta : c_{00} \to \mathbb{C}$$
$$s \mapsto \sum_{n=1}^{\infty} \frac{s_n}{n}.$$

Determine $(\ker \eta)^{\perp}$, the orthogonal complement of the kernel of η. Discuss whether a Riesz representation theorem is possible for the inner product space c_{00}.

(5) Let J_1, J_2, J_3 bounded self-adjoint operators on a Hilbert space $\mathcal{H}$, satisfying the commutation relations

$$[J_1, J_2] = iJ_3, \qquad [J_2, J_3] = iJ_1, \qquad [J_3, J_1] = iJ_2.$$

Define the operators

$$J_+ := J_1 + iJ_2, \qquad , J_- := J_1 - iJ_2, \qquad J^2 := J_1^2 + J_2^2 + J_3^2.$$

(a) Verify that $[J_\pm, J^2] = 0$ and $[J_3, J^2] = 0$.
(b) Verify that $[J_3, J_\pm] = \pm J_\pm$.
(c) Let $\psi \in \mathcal{H}$ be a simultaneous eigenvector for J^2 and J_3. Show that either $J_\pm\psi$ is a simultaneous eigenvector for J^2 and J_3, or $J_\pm\psi = 0$.
(d) Suppose that $\mathcal{H}$ admits an orthonormal basis comprising eigenvectors of J^2. Deduce that the eigenvalues of J_3 must be an integer multiple of $\frac{1}{2}$.
(e) Consider the following "orbital angular momentum" operators

$$L_1 = -i\Big(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\Big), \qquad L_2 = -i\Big(x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}\Big), \qquad L_3 = -i\Big(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\Big),$$

on the Schwartz space $\mathscr{S}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$. (The space of smooth functions whose partial derivatives of any order decay faster than any polynomial in the coordinates x_1, x_2, x_3). Verify that

$$[L_1, L_2] = iL_3, \qquad [L_2, L_3] = iL_1, \qquad [L_3, L_1] = iL_2.$$

Discuss whether or not we can apply part (d) to the operator L_3.

(6) Consider the three-parameter group

$$G = \left\{ \begin{pmatrix} 1 & b & y \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \,:\, b, y, s \in \mathbb{R} \right\},$$

called the Galilean group[1] (in one space and one time dimension). A time-dependent wavefunction is a map

$$\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C},$$
$$(t, x) \mapsto \Psi(t, x) \equiv \psi_t(x)$$

such that at every time t, the function $\psi_t : \mathbb{R} \to \mathbb{C}$ is in $L^2(\mathbb{R})$. Label the elements of G by (b, y, s). Fix $m > 0$, and represent the elements $(b, y, s) \in G$ as linear operators on $L^2(\mathbb{R})$, via the formula

$$\big(U_{(b,y,s)}\psi_t\big)(x) = e^{-imbx/\hbar}e^{imb^2t/2\hbar}\psi_{t-s}(x - y).$$

(a) Assuming that $U_{(0,0,s)}$ is unitary[2] for all $s \in \mathbb{R}$, explain why $U_{(b,y,s)}$ is a unitary operator on $L^2(\mathbb{R})$ for all $b, y, s \in \mathbb{R}$.
(b) Explain why $(b, y, s) \mapsto U_{(b,y,s)}$ does *not* define a group homomorphism (from G to the unitary operators on $L^2(\mathbb{R})$).

[1] The set of inertial frames of Newtonian space time is a torsor over G.
[2] This would be $e^{-iHs/\hbar}$ where H is the Schrödinger equation's Hamiltonian.