QUANTUM THEORY 2026
ASSIGNMENT 2 (DUE MONDAY 18 MAY 5PM)

(1) Consider the *unilateral* shift operator,

$$S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$
$$(x_1, x_2, x_3 \ldots) \mapsto (0, x_1, x_2, \ldots).$$

(a) What is the adjoint operator S^*?
(b) Is S normal?
(c) Determine the operator norm of S.
(d) Does S have any eigenvalues?
(e) Let $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Show that
  (i) λ is an eigenvalue of S^*.
  (ii) Is the range of $\lambda - S$ dense?
  (iii) Is λ an approximate eigenvalue of S?

(2) Let V be a Banach space, $B \in \mathcal{B}(V)$ be a bounded operator, and $T : \mathrm{Dom}(T) \to V$ be a closed linear operator. Show that TB and $T + B$ are closed operators.

(3) Let S, T and ST be densely-defined operators on a Hilbert space. Show that

$$T^* S^* \subset (ST)^*.$$

(4) A surjective map $C : \mathcal{H} \to \mathcal{H}$ is *antiunitary* if

$$\langle C\varphi | C\psi \rangle = \langle \psi | \varphi \rangle, \qquad \forall \varphi, \psi \in \mathcal{H}.$$

(a) Verify that an antiunitary C must be an antilinear map.
(b) Let H be a symmetric operator. Suppose there exists an antiunitary C such that
- $C^2 = 1_{\mathcal{H}}$ or $C^2 = -1_{\mathcal{H}}$
- C restricts to a map $\mathrm{Dom}(H) \to \mathrm{Dom}(H)$,
- $CH = HC$ on $\mathrm{Dom}(H)$.

Show that C maps $\mathcal{H}_\pm \to \mathcal{H}_\mp$, where $\mathcal{H}_\pm = \ker(H^* \mp i)$.

(5) Let $S^1 = \{e^{i\theta} : \theta \in [0, 2\pi)\}$ be the unit circle in the complex plane. Let $\Theta \in \mathcal{B}(L^2(S^1))$ be the bounded operator given by multiplication-by-θ,

$$(\Theta\psi)(\theta) = \theta\psi(\theta), \qquad \theta \in [0, 2\pi).$$

The angular momentum operator is the formally self-adjoint operator $L_z = -i\frac{d}{d\theta}$ on $C^\infty(S^1)$. For $n \in \mathbb{Z}$, we have

$$L_z \psi_n = n\psi_n,$$

where ψ_n is the normalized eigenfunction $\psi_n(e^{i\theta}) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$. Formally, we also have the commutator

$$[\Theta, L_z] = i.$$

Discuss what is wrong with the "paradoxical" computation

$$\begin{aligned} 0 \neq i\|\psi_n\|^2 = \langle \psi_n | i\psi_n \rangle = \langle \psi_n | [\Theta, L_z]\psi_n \rangle &= \langle \psi_n | \Theta L_z \psi_n \rangle - \langle \psi_n | L_z \Theta \psi_n \rangle \\ &= \langle \psi_n | \Theta L_z \psi_n \rangle - \langle L_z \psi_n | \Theta \psi_n \rangle \\ &= \langle \psi_n | \Theta n \psi_n \rangle - \langle n\psi_n | \Theta \psi_n \rangle = 0. \end{aligned}$$

*****PLEASE TURN OVER*****

(6) On the Hilbert space $L^2(\mathbb{R})$ (with Lebesgue measure on $\mathbb{R}$), consider the unitary translation operators U_t,

$$U_t\psi(x) = \psi(x-t), \qquad \psi \in L^2(\mathbb{R}),\, t \in \mathbb{R}.$$

Show that $t \mapsto U_t$ is a strongly-continuous 1-parameter unitary group.

(7) On the Hilbert space $L^2(S^1)$, let U_t be the rotation operators acting on $\psi \in L^2(S^1)$ as

$$U_t\psi(e^{i\theta}) = \psi(e^{i(\theta-t)}), \qquad e^{i\theta} \in S^1,\, t \in \mathbb{R}.$$

Also let V_s be the multiplication operators defined by

$$V_s\psi(e^{i\theta}) = e^{is\theta} \cdot \psi(e^{i\theta}), \qquad e^{i\theta} \in S^1,\, s \in \mathbb{R}.$$

Discuss whether U_t, V_s satisfy the Weyl commutation relations,

$$U_tV_s = e^{its}V_sU_t, \qquad \forall\, s,t \in \mathbb{R}.$$