QUANTUM THEORY 2026
ASSIGNMENT 3 (DUE MONDAY 15 JUNE 5PM)

(1) The *spin groups* in two and three dimensions are, respectively,

$$\mathrm{Spin}(2) = \left\{ \begin{pmatrix} w & 0 \\ 0 & \bar{w} \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \,:\, |w|^2 = 1 \right\},$$
$$\mathrm{Spin}(3) = \left\{ \begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \,:\, |w|^2 + |z|^2 = 1 \right\}.$$

Let S^2 be the 2-manifold of unit vectors in $\mathbb{R}^3$, and consider the surjective map

$$\pi : \mathrm{Spin}(3) \to S^2$$
$$\begin{pmatrix} w & -\bar{z} \\ z & \bar{w} \end{pmatrix} \mapsto \begin{pmatrix} 2\,\mathrm{Re}(\bar{w}z) \\ 2\,\mathrm{Im}(\bar{w}z) \\ |w|^2 - |z|^2 \end{pmatrix}.$$

The group Spin(2) acts on Spin(3) by right-multiplication.

(a) S^2 has the open cover $\{S^2 \setminus \{\mathrm{N}\}, S^2 \setminus \{\mathrm{S}\}\}$, where N is the north pole and S is the south pole.
   (i) Construct local trivializations to demonstrate that $\pi : \mathrm{Spin}(3) \to S^2$ is a principal Spin(2)-bundle.
   (ii) Work out the transition function for the above local trivializations.
   (iii) Briefly discuss whether $\pi : \mathrm{Spin}(3) \to S^2$ is globally trivializable.

(b) Elements of the Lie group SO(3) can be written as

$$(\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{a}) \in \mathrm{SO}(3),$$

where $\mathbf{u}, \mathbf{v}, \mathbf{a} \in \mathbb{R}^3$ are column vectors forming a positively-oriented orthonormal basis for $\mathbb{R}^3$.
Equip SO(3) with the projection map

$$\pi' : \mathrm{SO}(3) \to S^2$$
$$(\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{a}) \mapsto \mathbf{a},$$

and the action of the subgroup

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}(3) \right\}$$

by right-multiplication.
Briefly explain why $\pi' : \mathrm{SO}(3) \to S^2$ is a principal SO(2)-bundle.
[Hint: Think about the orthonormal frame bundle of S^2.]

(c) The *real* vector space $\mathrm{Herm}_0(2)$ of traceless complex Hermitian 2×2 matrices is spanned by the *Pauli spin matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   (i) Write $\mathrm{Tr}(-)$ for the trace of a square matrix (i.e. sum of diagonal entries). Show that

$$(A, B) \mapsto \frac{1}{2}\mathrm{Tr}(AB), \qquad A, B \in \mathrm{Herm}_0(2),$$

   defines an inner product, and that $\{\sigma_1, \sigma_2, \sigma_3\}$ is an orthonormal basis for $\mathrm{Herm}_0(2)$ with this inner product.
   (ii) Let $U \in \mathrm{Spin}(3)$. Define the 3×3 matrix $F(U)$ to have entries

$$F(U)_{ij} := \frac{1}{2}\mathrm{Tr}(\sigma_i U \sigma_j U^*), \qquad i, j = 1, 2, 3.$$

   Explain why $F(U)$ is an element of SO(3).
   [Hint: consider the square of elements of $\mathrm{Herm}_0(2)$.]

(iii) Verify that the map

$$F : \mathrm{Spin}(3) \to \mathrm{SO}(3)$$
$$U \mapsto F(U)$$

is a group homomorphism, and determine $\ker F$.

(iv) Check that F restricts to a homomorphism

$$\phi : \mathrm{Spin}(2) \to \mathrm{SO}(2).$$

(v) Deduce that

$$\pi = \pi' \circ F$$
$$F(p \cdot h) = F(p) \cdot \phi(h), \qquad \forall\, h \in \mathrm{Spin}(2), p \in \mathrm{Spin}(3).$$

[This shows that $\pi : \mathrm{Spin}(3) \to S^2$ is a ϕ-reduction of $\pi' : \mathrm{SO}(3) \to S^2$.]

(2) Let X be a smooth manifold. We defined the exterior derivatives $d : \Omega^0(X) \to \Omega^1(X)$ and $d : \Omega^1(X) \to \Omega^2(X)$ in the lectures.
Let $u, v \in \mathfrak{X}(X)$, $f, g \in \Omega^0(X)$, $\omega \in \Omega^1(X)$, and $h : Y \to X$ be smooth. Check the following properties:
   (a) $[f \cdot u, g \cdot v] = fg \cdot [u, v] + f(u(g)) \cdot v - g(v(f)) \cdot u$.
   (b) $h^*(g \cdot \omega) = (h^*(g)) \cdot h^*(\omega)$ and $h^*(dg) = d(h^*(g))$.
   (c) $d(f \cdot \omega) = df \wedge \omega + f \cdot d\omega$.
   (d) $d(df) = 0$.

(3) Let G be a Lie group. Show that for any $g \in G$, and any left-invariant vector field $v \in \mathfrak{g}$,
   - $(R_g)_* v \in \mathfrak{g}$;
   - $(R_g)_* v = \mathrm{Ad}_{g^{-1}}(v)$.

(4) Let G be a Lie group, $m : G \times G \to G$ be the group multiplication map, and $\phi : G \to G$ be the group inversion map. Calculate their derivatives at the identity element,

$$dm_{(e,e)} : T_eG \times T_eG \to T_eG, \qquad d\phi_e : T_eG \to T_eG.$$

Deduce that if G is Abelian, then its Lie algebra $\mathfrak{g}$ must have a trivial Lie bracket.

(5) Let $f : G \to H$ be a Lie group homomorphism, and $f_* : \mathfrak{g} \to \mathfrak{h}$ be the induced Lie algebra homomorphism. Show that the exponential map is natural, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{f_*} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{f} & H \end{array}.$$