Quantum Theory 2026

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1 Introduction

About this course. This is a mathematics course, meaning that almost everything is proof-based and rigorously defined. But we will always be guided by real physics. If you have already learned some quantum mechanics before, there might be

- A lot of functional analysis, or none at all;
- A lot of operator algebras, or none at all;
- A lot of abstract algebra and representation theory, or none at all;
- A lot of geometry and topology, or none at all;
- Modern (1960s onwards) experimental physics examples, or none at all.

This course is somewhat ambitious in the sense that we will try to cover a bit of everything mentioned above. I think that this is essential for a holistic understanding of quantum theory. Even if physics is just a hobby for you, I hope that you will at least learn how different branches of modern pure mathematics are actually quite unified.

History and need for new mechanics. Mechanics is about *motion* of *matter* in *space*. For example, Newtonian mechanics (17th century) has the ingredients:

- Space-time being identifiable with $\mathbb{R} \times \mathbb{R}^3$ (inertial frames), with standard metric (clocks, rulers, right angles).
- Motion of "point particle" being a path $\gamma : \mathbb{R} \to \mathbb{R}^3$.

- "Free motion" being straight paths with constant velocity.
- "Mass", or "inertia", being a property of the matter particle quantifying how it resists acceleration when placed in a "force" vector field,

$$\mathbf{F} = m\ddot{\gamma}$$

In practice, on a large enough scale, a "rigid body" (e.g. earth) is idealized as a single point at its centre-of-mass for certain purposes (e.g. orbital motion of planets around the sun). Zooming in, we have to take "internal motion" (e.g. spin rotation about the north-south axis) into account as well.

A typical example of $\mathbf{F}$ comes from Newton's gravitation theory. An (pointlike) object of mass M pulls an object of mass m towards it according to $F = \frac{GMm}{r^2}$, where r is the distance between the objects, and G is some universal constant. For two point masses, this can be solved for given initial positions and velocities for the two paths. But not for three point masses or more...

Basic question: How do we know how much "mass" an object has? The answer was embarrassingly primitive for a long time — one kept a local kilogram standard inside a vault in Paris, then duplicated it and propagated "identical" standard kilograms all over the world to calibrate other local weighing scales. In 2019, physicists finally found a better "global" universal solution. This involves Planck's universal constant from quantum mechanics, and some macroscopic quantum mechanical phenomena discovered in the late 21st century. By the end of the course, we should be able to appreciate how "quantum mechanical" the "classical" concept of mass actually is!

Besides mechanics of point particles, there is also wave mechanics. For example, some medium could be deformed from its equilibrium condition. The deformation data constitutes a "wave" — a time-dependent function $\psi(t; \mathbf{x})$ of the position $\mathbf{x}$ in the medium. Examples: water wave, sound waves, etc, where the value of ψ is the displacement (a vector quantity), and waves can be added and interfere ("superposition"). There are also "scalar"-valued ψ (e.g., temperature, pressure....). Such *classical* wave functions satisfy certain PDEs, *but only within certain regimes*. These PDEs usually involve some geometric principle, so operators like Laplacians often occur. Later, we will be more careful about what "vector", and "scalar" really means, because quantum mechanics forces us to use "spinor"-valued wavefunctions. We will

see that this concept comes from *spin geometry*, which is a refinement of classical Riemannian geometry.

Light presented a difficult puzzle as it seemed to exhibit properties of both waves and particles. Newton had a corpuscular (particle) theory of light, and it was usual to draw rays (paths) of light reflecting, refracting, etc. Danish astronomer Rømer also observed that light has a finite speed. In general, light was also observed to propagate in straight lines, so it seemed like something with no inertia (mass). In 1660s, Newton split a ray of white light into a *spectrum* of differently-coloured light rays, and speculated that there were different kinds of corpuscles of light corresponding to the differently coloured rays.

But in 1801, Young conducted the famous double-slit experiment, showing that light behaved like waves which diffracted away from the two slits, which subsequently superposed to form an interference pattern of alternating intensities on a screen. In the surrounding decades, the phenomenon of electricity, and its relationship with magnetism, was discovered. In particular, it was found that matter has a property, called *electric charge*, which in the presence of electricity and magnetism, experiences a certain force affecting its motion (in accordance with Newton's second law). Indeed, a pair of electric charges exerts forces on each other, much like in Newton's gravitational law. Crucially, this force could be *repulsive*, and so there seemed to be two types of charges, positive and negative, whereas mass is always positive. All of these properties were elegantly unified by Maxwell in the 1860s. There, it was proposed that there were electromagnetic vector fields subject to Maxwell's wave equations, and that a charged particle in such a field experiences a *Lorentz force*. Light was explained to be certain propagating wave solutions to Maxwell's equations, labelled by frequencies corresponding to different colours, all having the same speed of propagation. Besides visible light, there were also invisible infrared and ultraviolet radiation (lower and higher frequencies than visible light), and radio waves (very low frequency).

An importance source of electromagnetic radiation is a hot object (stars, candles, ovens, ...). Of particular interest is a (approximate) *black-body*, which absorbs/radiates electromagnetic radiation of all frequencies. At each fixed temperature, the observed intensity of black-body radiation as a function of frequency tended to diverge significantly from that predicted by existing theories at higher frequencies. In 1900, Planck proposed an alternative radiation law, by introducing a new concept which he later called "an act of

desperation". Specifically, he introduced the first *quantum postulate*: electromagetic radiation is emitted in *discrete amounts* with energy

$$E = h\nu,$$

where ν is the frequency, and h is "Planck's universal constant". The constant h has units of energy $\times$ time, and we note for later use that angular momentum, as well as "action" in analytic classical mechanics, has these units as well. Planck's postulate was controversial, since the prevailing understanding of light at that time was based on waves. But Planck's postulate allowed him to derive a much more accurate radiation law involving the new parameter h, and by curve fitting, he obtained a fairly accurate estimate of h which differs from the modern one by only about 1%.

A little earlier, in 1897, J.J. Thomson found that "cathode rays" were made up of charged matter particles, namely, *electrons*. It was also found that electrons could be ejected from a metal plate by shining light on a metal plate. Naïvely, higher intensity light should result in faster ejected electrons. Experimentally, however, it was found that emitted electron energy was independent of the intensity, and only occurred when the light frequency was above a certain threshold. In 1905, Einstein took the radical step of interpreting Planck's hypothesis literally, to explain this unusual *photoelectric effect*. He proposed that light, in its interaction with matter, behaved as though it comprises discrete quanta (called "photons"), and if the basic photon energy $E = h\nu$ falls below the threshold, electrons cannot be "freed" by absorbing such insufficiently energetic photons.

In subsequent years, more and more *macroscopic* experiments demonstrated this flavour of "quantization/discretization".

- In 1909, Millikan and Fletcher did the famous oil drop experiment establishing that electric charge occured in integer multiples of some "elementary charge" (that of an electron).

- Hydrogen gas was found to absorb/emit only certain frequencies of electromagnetic radiation. In accordance with $E = h\nu$, only certain discrete "energy jumps" seemed occur for the electrons in hydrogen atoms ("emission spectrum").

- In 1922, the Stern–Gerlach experiment demonstrated that silver atoms had spin angular momentum, which when measured along a given axis,

took on only two discrete values instead of a continuum of possible values.

In view of the electron-orbiting-nucleus picture of atoms being established at that time (Rutherford), Bohr postulated, in 1913, that the (orbital) angular momentum of electrons was somehow quantized to integer multiples of $\hbar = \frac{h}{2\pi}$, and consequently their possible energy levels would also be discretized, matching the discrete energy jumps deduced from the observed emission spectrum. In 1924, de Broglie interpreted Bohr's angular quantization condition as a standing wave condition, in the sense that the electron was (also) a wave, and its orbit had to exactly fit an integer number of wavelengths. It seemed that "matter particles" like electrons might sometimes be better modelled as waves. Indeed, in 1927, the electron version of the double slit experiment verified the phenomenon of electron wave interference. But what kind of wave are these "electron waves"? (How are their interference patterns "measured"?) What equation of motion governs it?

In 1925, Schrödinger proposed his famous wave equation,

$$i\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi, \tag{1.1}$$

to be solved for a complex-valued *wavefunction*

$$\psi \equiv \psi(t;\mathbf{x}) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}.$$

Here, m is the mass of the "quantum particle" whose "motion" is being studied, and $\Delta = \sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}$ is the Euclidean space Laplacian, while

$$V = V(t;\mathbf{x}) : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$$

is a real-valued *potential* function acting on $\psi(t;\mathbf{x})$ by pointwise multiplication. The variables t and $\mathbf{x}$ are time and spatial coordinates, respectively. Also, m is the mass of the "quantum particle" whose "state" is being modelled by the wavefunction ψ. In 1927, Pauli introduced a modification, which helped account for the quantization of spin angular momentum.

One can immediately do practical quantum mechanics by solving the Schrödinger equation for ψ with a given potential function V. Let us sketch this in the case of a time-independent $V \equiv V(\mathbf{x})$. Write $\psi_t(\mathbf{x}) \equiv \psi(t;\mathbf{x})$, and write the Schrödinger equation as

$$i\hbar\partial_t\psi_t = H\psi_t, \tag{1.2}$$

where

$$H = \underbrace{-\frac{\hbar^2}{2m}\Delta}_{\text{kinetic}} + \underbrace{V}_{\text{potential}} \tag{1.3}$$

is the total energy operator, called the *Hamiltonian*. Formally, the solution to (1.2) is

$$\psi_t = e^{-iHt/\hbar} \cdot \psi_0. \tag{1.4}$$

The difficulty is that H is not a number, but a linear operator, so one has to make sense of what $e^{-iHt/\hbar}$ means. In the special case where $\psi_0 = \psi_0^{(E)}$ is an eigenfunction of H with eigenvalue $E \in \mathbb{R}$, i.e.,

$$H\psi_0 = E\psi_0,$$

we can treat H as though it was just multiplication-by-E. The associated solution is

$$\psi_t^{(E)}(\mathbf{x}) = e^{-iEt/\hbar}\psi_0^{(E)}(\mathbf{x}), \tag{1.5}$$

and is called a *stationary state* of energy E. The general idea is that a general ψ_t is a linear combination of eigenfunctions of H, then each eigenfunction component of ψ_t evolves like (1.5), independently of other eigenfunction components. This could be formalized mathematically by saying that H admits a "nice" spectral decomposition.

For generally V, it is difficult to find the eigenfunctions of $H = -\frac{\hbar^2}{2m}\Delta + V$. But crucially, this can be done exactly, in the case of the *Coulomb potential* function

$$V(\mathbf{x}) = -\frac{e^2}{4\pi\varepsilon_0|\mathbf{x}|}.$$

Here e is the elementary charge of the electron (or minus that of the proton nucleus), and ε_0 is the permittivity of free space from classical electromagnetism. The Hamiltonian with this choice of V models the electron in a hydrogen atom. (The precise analysis of this eigenvalue problem will be done later in the course, and involves the difficult problem of proving essential self-adjointness of the Hamiltonian.)

According to the rules of the "new quantum mechanics", if the electron is in the stationary state whose $\psi_0^{(E)}$ has energy eigenvalue E, then a measurement of its energy will definitely yield E. A general state ψ, which is a complicated linear combination of such stationary states, does not have a definite energy at all. Instead, a measurement of energy will only ever have

eigenvalues of H as outcome. Furthermore, the *probability* of E being the measured outcome depends on the weight of $\psi_0^{(E)}$ appearing in the linear combination.

In particular, such rules mean that the jumps in energy eigenvalues are the only energy changes that can be observed for the electron. Indeed, the theoretically determined eigenvalue jumps for the Hamiltonian with Coulomb potential coincides with that determined empirically by the hydrogen emission spectrum. This was a key success of Schrödinger's new quantum wave mechanics.

We have mentioned that energy is a certain observable quantity that may be predicted from the wavefunction. But is the wave*function* itself something that can be measured, a priori?

A clue is provided by the observation that $\psi_t(\mathbf{x})$ has no physical units — it is just a *complex* number. It is not like a spatial displacement, or temperature, pressure, charge density, etc., which could be probed by the appropriate physical measuring instrument. In 1926, Born provided the probability interpretation of ψ. Informally, it is a "probability amplitude", in the sense that the $\mathbb{R}_{\geq 0}$-valued function $|\psi_t(-)|^2$ is the probability density function for the position, at time t. For this interpretation, we need the normalization

$$\int_{\mathbb{R}^3} |\psi_t(\mathbf{x})|^2 \, d\mathbf{x} = 1$$

to hold at all t. In other words, each $\psi_t(-)$ is a square integrable function on $\mathbb{R}^3$ with unit L^2-norm. Inspecting the formal evolution formula (1.4), we see that $e^{-iHt/\hbar}$ must be an operator which preserves the L^2-norm. (Precisely, $t \mapsto e^{iHt/\hbar}$ is a 1-parameter unitary group.)

In theory, if we put a (macroscopic) detector in a spatial region $B \subset \mathbb{R}^3$ to determine whether the particle "is" in a certain spatial region $B \subset \mathbb{R}^3$; the probability of detection (at time t), which is a binary yes/no outcome, would be

$$\int_B |\psi_t(\mathbf{x})|^2 \, d\mathbf{x} = \langle \psi_t | \chi_B \cdot \psi_t \rangle \in [0, 1].$$

Formally, the observable of "detection at B" corresponds to the linear operator of multiplication-by-χ_B, the indicator function for B. In the double-slit experiment for electrons, the screen is basically a collection of position detectors, and the interference pattern is basically a picture of $|\psi_t(\mathbf{x})|^2, \mathbf{x} \in$ screen.

In a similar way, the energy observable is the Hamiltonian linear operator H, and the expected value of an energy measurement when the quantum state is ψ_t is given by

$$\langle \psi_t | H\psi_t \rangle \in \mathbb{R}.$$

In particular, if $\psi_t = e^{-iEt/\hbar}\psi_0^{(E)}$ is an energy-E stationary state, we have

$$\langle \psi_t | H\psi_t \rangle = \langle \psi_0^{(E)} | H\psi_0^{(E)} \rangle = E \underbrace{\langle \psi_0^{(E)} | \psi_0^{(E)} \rangle}_{=1} = E,$$

independently of t. (Hence the terminology "stationary".) All these rules are very nicely formulated in the language of functional analysis and spectral theory of linear operators on Hilbert spaces.

So quantum mechanics is radically different from the old classical-type mechanics, where one studied trajectories (paths, or waves) in space with well-defined positions/velocities. Indeed, a lot of thought has been put into whether or not some kind of classical-style mechanics can account for all the accumulated experimental knowledge. The answer seems to be "no", and also, "why bother"?

The issue is that the classical concept of trajectories is already a mathematical idealization, as already emphasized by Riemann, in his famous 1854 lecture (which is better-known to mathematicians for introducing Riemannian manifolds), *Über die Hypothesen, welche der Geometrie zu Grunde liegen.*

> *. . . it seems that the empirical notions on which the metrical determinations of space are founded, the notion of a solid body and of a ray of light, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena. . .*

In other words, the concept of some (semi-)Riemannian/symplectic manifolds, geodesics, etc., which underlie classical mechanics, has no ultimate empirical basis. In fact, it is even falsified in some sense, in experiments which demonstrate *macrosopic* quantum tunnelling and energy quantization phenomena (Nobel Prize 2025).

Furthermore, we can only reliably answer questions corresponding to actual experiments that we can perform (at least in principle). As A. Peres puts beautifully,

> *Quantum phenomena do not occur in a Hilbert space. They occur in a laboratory.*

So, beware of grandiose statements like "the Hilbert space/wavefunction of the universe".

There is still a very subtle question regarding the necessity of *complex* wavefunctions, rather than real ones. In fact, one of the most important lessons from modern quantum mechanics (1960s onwards) is that the quantum mechanical "wavefunction" is, mathematically and conceptually speaking, not even a function at all! (For example, see the paradox discussed in Section 1.1.)

Rather, it is a section of a U(1)-line bundle with connection. Understanding the distinction between a section and a function is important if we want to account for experimental phenomena like the *Aharonov–Bohm effect*. This difference is clarified by the differential-geometric mathematics underlying *gauge theory*. The principles of gauge theory are foundational in modern differential geometry and physics, and is the theoretical basis for the *Standard Model* of particle physics ("high-energy physics"), as well as a lot of the interesting phenomena in condensed matter physics (modern "low energy" quantum physics).

1.1 Galilean invariance of Schrödinger equation?

For simplicity, we consider a one-dimensional space, and write the space-time coordinates as (t, x). It is important to realize that the coordinates (t, x) refer to a choice of inertial frame. As we know, Newtonian mechanics is Galilean invariant, meaning that there is no absolute rest frame. A "boosted" frame moving at velocity b with respect to a given inertial frame is another equally valid inertial frame. Let (t', x') be the space-time coordinates for this alternative inertial frame, i.e.,

$$(t', x') = (t, x - bt).$$

Now, if $\psi \equiv \psi_t(x)$ is a complex-valued *function* on space-time when expressed in coordinates with respect to the first inertial frame, then in coordinates with

respect to the alternative inertial frame, the *same* function would have the form $\psi' \equiv \psi'_{t'}(x')$, where

$$\psi' \equiv \psi'_{t'}(x') = \psi(t, x). \tag{1.6}$$

Let us investigate how the Schrödinger wave equation behaves under different choices of inertial frames. In the original (t, x) coordinates, the *free Hamiltonian* ($V \equiv 0$) is

$$H_{\text{free}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}, \tag{1.7}$$

acting on (smooth, square-integrable) wavefunctions $\psi_t : \mathbb{R} \to \mathbb{C}$. Suppose ψ_t satisfies the Schrödinger equation for the free Hamiltonian,

$$i\hbar\frac{\partial}{\partial t}\psi_t = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_t. \tag{1.8}$$

Since we could have chosen to use (t', x') coordinates instead, we should expect that $\psi'_{t'}$ likewise satisfies

$$i\hbar\frac{\partial}{\partial t'}\psi'_{t'} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}\psi'_{t'}. \tag{1.9}$$

But this is *false* if we use formula (1.6) for ψ'! In other words, the Schrödinger equation is incompatible with Galilean relativity if we insist that ψ transforms like an ordinary function!

The problem is that the partial derivative operators refer to coordinates, and have to be transformed accordingly as well. With $x = x' + bt'$ and $t = t'$, we have

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + b\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial x'} = \frac{\partial}{\partial x}.$$

Plugging these into (1.8) shows that ψ' solves a differential equation to (1.9). In other words, the form of the free Schrödinger equation looks different in different inertial frames! This is a serious problem, since it implies that there is some preferred inertial frame for which the free Schrödinger equation has the simple form (1.8), violating the Galilean relativity principle.

Here is a resolution to this problem. With a slight abuse of notation, write $U_{b;t}$ for the unitary operator of multiplication by the U(1)-valued function

$$u_{b;t}(x) = e^{-imbx/\hbar}e^{imb^2t/2\hbar}. \tag{1.10}$$

We claim that the naïve transformation rule (1.6) should be replaced by the rule

$$\psi'_{t'}(x') := (U_{b;t}\psi_t)(x). \tag{1.11}$$

Indeed, we calculate

$$\begin{aligned} i\hbar\frac{\partial}{\partial t'}\psi'_{t'} &= i\hbar\left(\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}\right)(U_{b;t}\psi_t) \\ &= i\hbar U_{b;t}\left(\frac{\partial}{\partial t} + \frac{imb^2}{2\hbar} + b\frac{\partial}{\partial x} - \frac{imb^2}{\hbar}\right)\psi_t \\ &= U_{b;t}\left(i\hbar\frac{\partial}{\partial t} + ib\hbar\frac{\partial}{\partial x} + \frac{mb^2}{2}\right)\psi_t, \end{aligned} \tag{1.12}$$

as well as

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}\psi'_{t'} &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(U_{b;t}\psi_t) \\ &= -\frac{\hbar^2}{2m}\frac{\partial}{\partial x}U_{b;t}\left(\frac{\partial}{\partial x} - \frac{imb}{\hbar}\right)\psi_t \\ &= -\frac{\hbar^2}{2m}U_{b;t}\left(\frac{\partial}{\partial x} - \frac{imb}{\hbar}\right)^2\psi_t \\ &= U_{b;t}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + ib\hbar\frac{\partial}{\partial x} + \frac{mb^2}{2}\right)\psi_t. \end{aligned} \tag{1.13}$$

Combining Eq. (1.12)–(1.13), we get

$$\left(i\hbar\frac{\partial}{\partial t'} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}\right)\psi'_{t'} = \left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi_t,$$

showing that ψ_t satisfies (1.8) iff $\psi'_{t'}$ satisfies (1.9). Thus, the (free) Schrödinger is compatible with the Galilean relativity principle, provided the "wavefunctions" transform according to the modified rule (1.11) when the inertial frame is boosted by velocity b.

The transformation rule (1.11), highlights a crucial fact. Superficially, one notes that the remedy does not work if ψ_t is *real*-valued. More importantly, it shows that ψ_t cannot be interpreted as an ordinary function! Rather, ψ is really a section of a Hermitian line bundle $\mathcal{L} \to \mathbb{R} \times \mathbb{R}$, with each ψ_t an L^2-section of $\mathcal{L}_t \to \{t\} \times \mathbb{R}$ (the part of the bundle lying over the spatial slice at time t). In order to describe the section ψ as a function, we have to pick

a fiducial "gauge". This is exactly analogous to how a tangent vector field is describable as an $\mathbb{R}^d$-valued function, only after picking some reference frame field. The operator $U_{b;t}$ of multiplication by $u_{b;t}$ implements a *gauge transformation*, changing the reference gauge and thus the representation of ψ_t as a function.

Furthermore, there is no preferred reference gauge, so the functional representation of ψ (which is gauge-dependent) cannot be measured. Only quantities which are independent of gauge choice can have physically measurable consequences. For example, the probability density function $|\psi_t(-)|$ is invariant under the application of gauge transformations such as $U_{b;t}$. In contrast, the gauge-dependent function $\psi_t(-)$ can never be measured!

Exercise 1.1. We can also spatially translate (the origin of) an inertial frame by some $y \in \mathbb{R}$, to obtain another inertial frame. Boosting by velocity b, then spatially translating by y, results in the same inertial frame as spatially translating by y, then boosting by velocity b. Let $\psi_0(-)$ be a wavefunction at time $t = 0$ (with respect to some initial inertial frame and gauge). For a fixed mass $m > 0$, let

$$(S_b \cdot \psi_0)(x) := (U_{b;0}\psi_0)(x) = e^{-imbx/\hbar}\psi_0(x),$$

and

$$(L_y \cdot \psi_0)(x) := \psi_0(x - y)$$

be the transformed wavefunctions at $t = 0$, corresponding to the boost and the spatial translation, respectively. Verify that the linear operators S_b and L_y do *not* commute, but rather,

$$S_b L_y S_b^{-1} L_y^{-1} = e^{-imby/\hbar}. \tag{1.14}$$

Eq. (1.14) says that the symmetry group of the Schrödinger equation for mass m is not the classical *Galilean group* of transformations between inertial frames, but rather, a certain centrally extended group called the *Bargmann group*. Mathematically, there is an exact sequence

$$1 \to \mathrm{U}(1) \to \text{Bargmann} \to \text{Galilean} \to 1,$$

which does not split. The failure to split is captured by the m-dependent phase factor $e^{-imby/\hbar}$ in (1.14). This also indicates a very close relationship between mass in quantum mechanics and Planck's constant. You are encouraged to read up on the internet about how the kilogram is now defined in

terms of h. The U(1) is important for coupling electromagnetism to charged quantum matter (e.g., to describe electrons in electromagnetic fields), as we will learn.

There was another revolutionary idea of Einstein in 1905, slightly predating the quantum revolution. This is the *special* relativity principle, which basically says that space-time does *not* have the Euclidean $\mathbb{R} \times \mathbb{R}^3$ metric structure with Galilean velocity boosts of inertial frames. Instead, spacetime has a semi-Riemannian metric, with "temporal" and "spatial" coordinates having the equal footing except for a sign difference in the "spacetime metric", and the speed of light c as a conversion factor. The Galilean velocity boosts are replaced by the so-called *Lorentz* boosts.

The Schrödinger equation is certainly *not* compatible with Einstein's special relativity principle, since it is first order in t and second order in x. In practice, this means that when the velocities involved are close to the speed of light, the Schrödinger equation will fail to a good quantum mechanics model. The construction of a quantum mechanical wave equation compatible with Einstein's special relativity would soon be carried out by Dirac, in 1928. We will learn aboutthe famous *Dirac equation* and the concept of *spinors* later in the course.