12 Principal bundles and gauge

12.1 Frame bundle, informally

In Section 2.1, we explained that a basis for an n-dimensional vector space V is a linear isomorphism $\beta : \mathbb{K}^n \to V$. We shall also call a basis a *frame*. The set of frames for V is a right $\mathrm{GL}(n)$-torsor,

$$\begin{aligned} \{\text{Frames}\} \times \mathrm{GL}(n) &\to \{\text{Frames}\} \\ (\beta, g) &\mapsto \beta \circ g. \end{aligned}$$

Only after fixing a reference basis β_{ref}, do we have identifications

$$V \cong \mathbb{K}^n, \qquad \{\text{Frames}\} \cong \mathrm{GL}(n), \tag{12.1}$$

with a frame β being identified with the $\mathrm{GL}(n)$ matrix g such that $\beta = \beta_{\text{ref}} \circ g$.

Similarly, if V is a real (resp. complex) inner product space, the set of orthonormal frames is a right $\mathrm{O}(n)$-torsor (resp. $\mathrm{U}(n)$).

Now consider a vector bundle E of rank n. Each fibre E_x has a space of frames, denoted $\{\text{Frames}\}_x$, which is a $\mathrm{GL}(n)$-torsor as described above. Altogether, there is a "frame bundle" over X,

$$\mathrm{Fr}(E) = \bigsqcup_{x \in X} \{\text{Frames}\}_x, \tag{12.2}$$

with each fibre being a $\mathrm{GL}(n)$-torsor. The precise construction of $\mathrm{Fr}(E)$, provided in Section 12.3.3 later, exhibits it as a "principal $\mathrm{GL}(n)$-bundle".

Similarly, if E is equipped with a smoothly-varying inner product on its fibres, then there is an "orthonormal frame bundle" $\mathrm{Fr}^{\mathrm{O}}(E)$, which is a principal $\mathrm{O}(n)$ bundle with fibres being $\mathrm{O}(n)$-torsors.

12.2 Lie groups

Definition 44. A *Lie group* G is a manifold with a group structure such that

$$G \times G \to G, \qquad (g_1, g_2) \mapsto g_1 g_2^{-1}$$

is smooth.

Thus the group operations (inversion and composition) are smooth maps. In partigular, for each $g \in G$, *left multiplication* by g and *right multiplication* by g,

$$L_g : g' \mapsto gg', \qquad R_g : g' \to g'g, \qquad g' \in G,$$

are diffeomorphisms $G \to G$.

Example 12.1. The group of invertible matrices $\mathrm{GL}(n;\mathbb{R})$ is an open (why) subset of the vector space of matrices $\mathrm{M}_n(\mathbb{R})$, thus it is a manifold. The group operation is matrix multiplication, which is polynomial in the standard entrywise coordinates, thus smooth. So $\mathrm{GL}(n;\mathbb{R})$ is a Lie group, and similarly for $\mathrm{GL}(n;\mathbb{C})$.

The tangent spaces are

$$T_g\mathrm{GL}(n;\mathbb{R}) = T_g \underbrace{\mathrm{M}_n(\mathbb{R})}_{\text{manifold}} = \underbrace{\mathrm{M}_n(\mathbb{R}) = \mathbb{R}^{n^2}}_{\text{vector space}}.$$

Thus the tangent bundle of $\mathrm{GL}(n;\mathbb{R})$ is a canonically trivial vector bundle of rank $n^2 = \dim \mathrm{GL}(n;\mathbb{R})$. Similarly for $\mathrm{GL}(n;\mathbb{C})$.

The subgroup with positive determinant is denoted $\mathrm{GL}(n,\mathbb{R})^+$, and is likewise a Lie group.

More generally, for an n-dimensional vector space V, the invertible linear maps $\mathrm{GL}(V)$ are identifiable with $\mathrm{GL}(n;\mathbb{K})$ by picking any basis for V. Such an identification can be viewed as a coordinate chart, making $\mathrm{GL}(V)$ a manifold, thus a Lie group.

Exercise 12.1. Consider $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and the linear subspace $\mathrm{Sym}(n)$ of symmetric matrices. Check that 1_n is a regular value for the map

$$f : \mathrm{M}_n(\mathbb{R}) \to \mathrm{Sym}(n), \qquad A \mapsto A^{\mathrm{t}}A.$$

Example 12.2. Exercise 12.1 and Prop. 11.3 show that $\mathrm{O}(n) = f^{-1}(1_n)$ is a submanifold of $\mathrm{M}_n(\mathbb{R})$ (and also a submanifold of $\mathrm{GL}(n,\mathbb{R})$). It follows (by Exercise 11.6) that $\mathrm{O}(n)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$. Similarly, $\mathrm{SO}(n)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})^+$, while $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are Lie subgroups of $\mathrm{GL}(n,\mathbb{C})$.

12.3 Principal bundle definitions

12.3.1 General principal bundles

Definition 45. Let G be a Lie group. The *trivial principal G-bundle* over a manifold X is the product manifold $X \times G$, equipped with the projection

map

$$\pi_X : X \times G \to X, \qquad (x, g) \mapsto x,$$

and the fibrewise right G-action,

$$(x, g') \cdot g = (x, g'g), \qquad x \in X,\ g', g \in G.$$

Definition 46. Let G be a Lie group. A *principal G-bundle* over a manifold X is a manifold P with a smooth surjection $\pi : P \to X$ and a smooth right G-action (by diffeomorphisms),

$$P \times G \to P, \qquad (p, g) \mapsto p \cdot g,$$

such that

- The G-action restricts to fibres,

$$\pi(p \cdot g) = \pi(p), \qquad p \in P,\ g \in G;$$

- Each $x \in X$ is contained in some open neighbourhood U such that there exists a diffeomorphism

$$\Phi : P|_U := \pi^{-1}(U) \to U \times G,$$

satisfying

$$\begin{aligned} \pi_U \circ \Phi &= \pi \\ \Phi(p \cdot g) &= \Phi(p) \cdot g, \qquad p \in \pi^{-1}(U), g \in G. \end{aligned}$$

Definition 47. Let $\pi : P \to X$ be a principal G-bundle. In Definition 46, the map Φ is called a *local trivialization*. A *bundle atlas* is a collection of local trivializations

$$\Phi_\alpha : P|_{U_\alpha} \stackrel{\cong}{\to} U_\alpha \times G, \qquad \alpha \in \mathcal{I},$$

such that $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ covers X.

As with Exercise 11.8, the projection map π of a principal G-bundle can be checked to be a submersion. By Corollary 11.5, we can, for example, restrict a local trivialization further to $\pi^{-1}V$ for some submanifold $V \subset U$, and obtain another local trivialization $P|_V \cong V \times G$. As usual, we allow compatible local trivializations to be added to a bundle atlas, and formally pass to the maximal bundle atlas. Here, compatibility means smoothness of transition functions, as described in Section 12.3.2.

Exercise 12.2. Let P be a principal G-bundle. Check that the following hold:

- For any $p \in P$, its G-orbit is precisely the fibre containing p.
- The right G-action on P is free (every non-identity element $g \in G$ has no fixed points).

In a trivial principal G-bundle, each fibre is regarded as a G-torsor with basepoint, thus all fibres are identified with G. In a general principal G-bundle, each fibre is a G-torsor *without* basepoint (Exercise 12.2). A local trivialization specifies a local convention for identifying the various fibres as G (i.e., G-torsors with basepoint).

Example 12.3. Recall the tautological line bundle $\mathcal{L}^{\mathbb{R}} \to \mathbb{RP}^1$. For each $\ell \in \mathbb{RP}^1$, the fibre $\mathcal{L}^{\mathbb{R}}_\ell$ contains two unit vectors. A choice of such a unit vector is equivalently a choice of orthonormal frame for $\mathcal{L}^{\mathbb{R}}_\ell$.

The *unit sphere bundle* $S(\mathcal{L}^{\mathbb{R}})$ is the subset of $\mathcal{L}^{\mathbb{R}}$ comprising the unit-length vectors. In fact, $S(\mathcal{L}^{\mathbb{R}})$ is the bundle of orthonormal frames of $\mathcal{L}^{\mathbb{R}}$. The group $\mathrm{O}(1)$ acts on $S(\mathcal{L}^{\mathbb{R}})$ by swapping the two possible orthonormal frames at each $\ell \in \mathbb{RP}^1$. This gives $S(\mathcal{L}^{\mathbb{R}})$ the structure of a principal $\mathrm{O}(1)$-bundle over $\mathbb{RP}^1$. (Work out the details as an exercise.)

Interestingly, $S(\mathcal{L}^{\mathbb{R}})$ is not two disconnected copies of $\mathbb{RP}^1$. Rather, $S(\mathcal{L}^{\mathbb{R}})$ can be shown to be diffeomorphic to a circle S^1; it is a *connected* double cover of $\mathbb{RP}^1$. Therefore, $S(\mathcal{L}^{\mathbb{R}})$ is a non-trivializable principal $\mathrm{O}(1)$-bundle.

Example 12.4. Write $\mathbb{CP}^1$ for the space of one-dimensional complex subspaces of $\mathbb{C}^2$. It may be shown that $\mathbb{CP}^1$ is diffeomorphic to a 2-sphere. The complex tautological line bundle $\pi : \mathcal{L}^{\mathbb{C}} \to \mathbb{CP}^1$ is defined analogously to the real case. For each $\ell \in \mathbb{CP}^1$, the fibre $\mathcal{L}^{\mathbb{C}}_\ell$ contains a circle of unit vectors, and a choice of unit vector is equivalent to a choice of orthonormal frame for the fibre. The unit sphere bundle $S(\mathcal{L}^{\mathbb{C}})$ is thus the orthonormal frame bundle for $\mathcal{L}^{\mathbb{C}}$. The circle group $\mathrm{U}(1)$ acts on $S(\mathcal{L}^{\mathbb{C}})$ by "rotating orthonormal frames" at each ℓ. This makes $\pi : S(\mathcal{L}^{\mathbb{C}}) \to \mathbb{CP}^1$ a principal $\mathrm{U}(1)$-bundle (Exercise).

Interestingly, $S(\mathcal{L}^{\mathbb{C}})$ can be shown to be diffeomorphic to S^3, thus it is not diffeomorphic to $S^2 \times \mathrm{U}(1)$. So $S(\mathcal{L}^{\mathbb{C}})$ is a non-trivializable principal $\mathrm{U}(1)$-bundle.

12.3.2 Transition functions

Given two local trivializations,

$$\Phi_\alpha : P|_{U_\alpha} \to U_\alpha \times G$$
$$\Phi_\beta : P|_{U_\beta} \to U_\beta \times G,$$

we get two conventions for labelling the fibres over $x, x \in U_\alpha \cap U_\beta$. We have

$$\begin{aligned} \Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times G &\to (U_\alpha \cap U_\beta) \times G \\ (x, e) &\mapsto (x, g_{\alpha\beta}(x)), \qquad g_{\alpha\beta}(x) \in G \\ (x, g) = (x, e) \cdot g &\mapsto (x, g_{\alpha\beta}(x) g). \end{aligned} \tag{12.3}$$

The formula for (x, g) follows from that for (x, e) by G-equivariance, and the smooth function

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G, \tag{12.4}$$

is called a *transition function*.

Given a bundle atlas $\{\Phi_\alpha\}_{\alpha\in\mathcal{I}}$, we get a collection of transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{I}}$, each defined on a double overlap. It is clear that

$$g_{\alpha\alpha} = 1, \qquad g_{\beta\alpha} = g_{\alpha\beta}^{-1}, \qquad \alpha, \beta \in \mathcal{I}. \tag{12.5}$$

Furthermore, on triple overlaps, the associativity of group multiplication implies that

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \qquad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma, \quad \alpha, \beta, \gamma \in \mathcal{I}. \tag{12.6}$$

Eq. (12.6) is called the *cocycle condition*.

12.3.3 Frame bundle of a vector bundle as a principal bundle

Let us return to the frame bundle $\pi : \mathrm{Fr}(E) \to X$, which as a set was given by Eq. (12.2). It has a fibrewise right $\mathrm{GL}(n)$-action. We elaborate on the smooth aspects of its principal $\mathrm{GL}(n)$-bundle structure, and compare the transition functions for E with the transition functions for $\mathrm{Fr}(E)$.

Let $\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in\mathcal{I}}$ be a bundle atlas for E. If we restrict $\Phi_\alpha^{-1} : U_\alpha \times \mathbb{K}^n \xrightarrow{\cong} E|_{U_\alpha}$ to $\{x\} \times \mathbb{K}^n$, we get a linear isomorphism

$$\left(\Phi_\alpha|_{E_x}\right)^{-1} : \{x\} \times \mathbb{K}^n \to E_x, \qquad x \in U_\alpha.$$

In other words, $\left(\Phi_\alpha|_{E_x}\right)^{-1}$ is a basis/frame at x. Any other frame at x is uniquely specified by (precomposition with) $g \in \mathrm{GL}(n)$. Let us define the G-equivariant maps

$$\begin{aligned}&\widetilde{\Phi}_\alpha : \mathrm{Fr}(E)|_{U_\alpha} \to U_\alpha \times \mathrm{GL}(n)\\ &\underbrace{\left(\Phi_\alpha|_{E_x}\right)^{-1} \cdot g}_{\text{a frame at } x} \mapsto (x, g), \qquad \alpha \in \mathcal{I}.\end{aligned}$$

These $\widetilde{\Phi}_\alpha$ will serve as local trivializations of $\mathrm{Fr}(E)$ after $\mathrm{Fr}(E)$ is given a smooth structure. Furthermore, since $\Phi_\beta|_{E_x} \circ \Phi_\alpha|_{E_x}^{-1}$ is matrix-multiplication by $g_{\beta\alpha}(x)$ (by definition, see Eq. (11.9)), we have

$$\widetilde{\Phi}_\beta \circ \widetilde{\Phi}_\alpha^{-1} : (x, g) \mapsto (x, g_{\beta\alpha}(x)g), \qquad x \in U_\alpha \cap U_\beta, g \in G. \tag{12.7}$$

Thus, the transition functions for $\mathrm{Fr}(E)$ are the same as those for E.

For the smooth structure, we provide a sketch. Topologize $\mathrm{Fr}(E)$ by declaring $V \subset \mathrm{Fr}(E)$ to be open iff $V \cap \mathrm{Fr}(E)|_{U_\alpha}$ is open for all $\alpha \in \mathcal{I}$. In particular, the $\widetilde{\Phi}_\alpha : \mathrm{Fr}(E)|_{U_\alpha} \to U_\alpha \times \mathrm{GL}(n)$ become homeomorphisms, and they can be viewed as local coordinate charts making $\mathrm{Fr}(E)$ a topological manifold. (More precisely, we would compose $\widetilde{\Phi}_\alpha$ with some coordinate charts for U_α and $\mathrm{GL}(n)$ to get a coordinate chart for $\mathrm{Fr}(E)$.) The smoothness of coordinate changes is inherited from smoothness of $g_{\beta\alpha}$. So $\mathrm{Fr}(E)$ is a smooth manifold, with smooth right $\mathrm{GL}(n)$-action, and the $\widetilde{\Phi}_\alpha, \alpha \in \mathcal{I}$ become a bundle atlas of local trivializations for $\mathrm{Fr}(E)$.

12.4 Morphisms of principal bundles

Definition 48. A morphism between two principal G-bundles $P_1 \to X_1$ and $P_2 \to X_2$, is a smooth map $F : P_1 \to P_2$ which is G-equivariant, i.e.,

$$F(p \cdot g) = F(p) \cdot g, \qquad p \in P_1,\ g \in G.$$

Exercise 12.3. Let F be a morphism of principal G-bundles (Definition 48). Check that:

- For any $p \in P$, F restricts to a diffeomorphism between the fibre containing p and the fibre containing $F(p)$. Thus F induces a well-defined

map $f : X_1 \to X_2$ such that the following map commutes

$$\begin{array}{ccc} P_1 & \xrightarrow{F} & P_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow{f} & X_2 \end{array} . \tag{12.8}$$

- If F induces a diffeomorphism $f : X_1 \to X_2$, then F is itself a diffeomorphism, and $F^{-1} : P_2 \to P_1$ is also a morphism of principal G-bundles.

Definition 49. Two principal G-bundles P_1, P_2 over the same manifold X are *isomorphic*, if there is a G-equivariant diffeomorphism $F : P_1 \to P_2$ inducing the identity map id_X.

Example 12.5. Let $\pi : P \to X$ be a principal G-bundle, and $Z \subset X$ be a submanifold. The subset $P|_Z := \pi^{-1}(Z)$ is a submanifold of P (why?), and is itself a principal G-bundle. The inclusion $P|_Z \hookrightarrow P$ is a principal G-bundle morphism, inducing the inclusion $Z \hookrightarrow X$.

12.5 Principal bundle via transition functions

To what extent is a principal bundle determined by the data of its transition functions?

A priori, $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{I}}$ depends on a choice of bundle atlas. We can always modify the Φ_α to

$$\Phi'_\alpha = (\mathrm{id}_{U_\alpha} \times \lambda_\alpha^{-1}) \circ \Phi_\alpha,$$

by picking smooth functions

$$\lambda_\alpha : U_\alpha \to G, \qquad \alpha \in \mathcal{I}.$$

Then, with respect to the modified bundle atlas $\{\Phi'_\alpha\}_{\alpha\in\mathcal{I}}$, we get the modified transition functions $\{g'_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{I}}$, where

$$g'_{\alpha\beta} = \lambda_\alpha^{-1} g_{\alpha\beta} \lambda_\beta. \tag{12.9}$$

Nothing has happened to P. So when two collections of transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{I}}$ and $\{g'_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{I}}$ are related by Eq. (12.9), we should both collections to describe the same principal bundle. Indeed, the unprimed and primed atlases are compatible, so they would be contained in the same maximal bundle atlas.

Next, recall that an open cover $\{U_{\alpha'}\}_{\alpha' \in \mathcal{I}'}$ of X is a refinement of $\{U_\alpha\}_{\alpha \in \mathcal{I}}$, if each $U_{\alpha'}$ is contained in some U_α. By restricting Φ_α from $\pi^{-1}(U_\alpha)$ to $\pi^{-1}(U_{\alpha'})$ (recalling that the latter is a submanifold of P, why?), we get an alternative bundle atlas of local trivializations $\Phi_{\alpha'}$, thus another collection of transition functions,

$$g_{\alpha'\beta'} : U_{\alpha'} \cap U_{\beta'} \to G, \qquad \alpha', \beta' \in \mathcal{I}.$$

Again, nothing has happened to P.

Now suppose we have two principal bundles $P \to X$ and $P' \to X$, with respective bundle atlases of local trivializations $\Phi_\alpha, \Phi'_{\alpha'}$ and respective collections of transition functions

$$\begin{aligned} g_{\alpha\beta} &: U_\alpha \cap U_\beta \to G, & \alpha, \beta &\in \mathcal{I}, \\ g'_{\alpha'\beta'} &: U_{\alpha'} \cap U_{\beta'} \to G, & \alpha', \beta' &\in \mathcal{I}'. \end{aligned}$$

There is a common refinement of the open covers $\{U_\alpha\}_{\alpha \in \mathcal{I}}, \{U_{\alpha'}\}_{\alpha' \in \mathcal{I}'}$,

$$\{U_\alpha \cap U_{\alpha'}\}_{\alpha \in \mathcal{I}, \alpha' \in \mathcal{I}'},$$

and we can restrict the $g_{\alpha\beta}$ and $g'_{\alpha'\beta'}$ to double overlaps of this refined cover. The primed transition functions can now be compared with the unprimed transition functions.

Lemma 12.1. *Let $P \to X$ and $P' \to X$ be two principal G-bundles, with respective transition functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ defined with respect to the same open cover $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ of X. Then P and P' are isomorphic iff there exist smooth functions $\lambda_\alpha : U_\alpha \to G$ such that*

$$g'_{\alpha\beta}(x) = \lambda_\alpha(x)^{-1} g_{\alpha\beta}(x) \lambda_\beta(x), \qquad x \in U_\alpha \cap U_\beta,\ \alpha, \beta \in \mathcal{I}. \tag{12.10}$$

Proof. The transition functions $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are associated, respectively, to local trivializations

$$\Phi_\alpha : P|_{U_\alpha} \to U_\alpha \times G, \qquad \Phi'_\alpha : P'|_{U_\alpha} \to U_\alpha \times G.$$

The map $(\mathrm{id}_{U_\alpha} \times \lambda_\alpha)$ is an automorphism of the trivial bundle $U_\alpha \times G$. So

$$(\Phi'_\alpha)^{-1} \circ (\mathrm{id}_{U_\alpha} \times \lambda_\alpha) \circ \Phi_\alpha : P|_{U_\alpha} \to P'|_{U_\alpha} \tag{12.11}$$

is an isomorphism. Check, using (12.10), that

$$(\Phi'_\beta)^{-1} \circ (\mathrm{id}_{U_\beta} \times \lambda_\beta) \circ \Phi_\beta : P|_{U_\beta} \to P'|_{U_\beta}$$

is coincides with the isomorphism (12.11) over $P|_{U_\alpha \cap U_\beta}$. This ensures that we can consistently patch together all the local bundle isomorphisms (12.11) to get a well-defined global bundle isomorphism $P \to P'$. □

Remark. A collection of functions $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{I}}$ satisfying conditions (12.5)–(12.6) provide the data for patching a collection $\{U_\alpha \times G\}_{\alpha\in\mathcal{I}}$ of trivial principal bundles into a global principal bundle $\pi : P \to X$. Sketch: as a set, P is the quotient of

$$\bigsqcup_{\alpha\in\mathcal{I}} U_\alpha \times G \times \{\alpha\}$$

by the equivalence relation

$$(x, g, \alpha) \sim (x', g', \beta) \quad \text{iff} \quad x = x' \ \text{ and } \ g' = g_{\beta\alpha} g.$$

For full details of the topology, manifold structure etc., see Naber's Topology, Geometry and Gauge Fields, or Kobayashi–Nomizu's Foundations of Differential Geometry.

12.6 Local gauges and trivializations

When studying vector bundles, we saw that local trivializations correspond to local frames, i.e., local sections of the frame bundle. For a general principal bundle, the analogue of a local frame is usually called a "local gauge".

Definition 50. Let $\pi : P \to X$ be a principal G-bundle. A *local gauge* over an open neighbourhood $U \subset X$ is a smooth section over U, i.e., a smooth map $s : U \to P$ such that $\pi \circ s = \mathrm{id}_U$. A *global gauge* is a smooth section over X.

The trivial principal bundle $X \times G$ admits a globally defined *trivial gauge*,

$$s_{\mathrm{triv}} : x \mapsto (x, e),$$

where $e \in G$ is the identity element. A general principal bundle may not admit any global gauge.

If $\pi : P \to X$ is trivializable, then a choice of global trivialization $\Phi : P \to X \times G$ can be used to convert s_{triv} into a global gauge for P, via

$$s_\Phi := \Phi^{-1} \circ s_{\text{triv}} : x \mapsto \Phi^{-1}(x, e). \tag{12.12}$$

Conversely, a global gauge s for P determines a global trivialization as follows (thus P is trivializable). At each $x \in X$, s picks out a basepoint $s(x)$ for the fibre P_x. Since the fibre P_x is a G-torsor, its elements can be uniquely labelled as

$$s(x) \cdot g, \qquad g \in G.$$

Therefore, we obtain a bijection

$$\Phi_s : P \to X \times G, \qquad s(x) \cdot g \mapsto (x, g), \tag{12.13}$$

which is easily checked to be G-equivariant. With some general smooth manifold and Lie group theory, one verifies that Φ_s is a diffeomorphism (see Lemma 4.2.7 of Hamilton's Mathematical Gauge Theory book for details), thus it gives a global trivialization of P.

From the $S(\mathcal{L}^{\mathbb{C}})$ and $S(\mathcal{L}^{\mathbb{R}})$ examples, we know that non-trivializable principal bundles exist. So in general, only *local* gauges (or local trivializations) are available. Suppose $\Phi : P|_U \to U \times G$ is a local trivialization. Then, as in the global case (12.12), we have the associated local gauge

$$\begin{aligned} s_\Phi : U &\to P|_U \\ x &\mapsto \Phi^{-1}(x, e). \end{aligned} \tag{12.14}$$

Conversely, as in (12.13), a local gauge $s : U \to P|_U$ determines the local trivialization

$$\begin{aligned} \Phi_s : P|_U \equiv \pi^{-1}(U) &\to U \times G, \\ (s(x) \cdot g) &\mapsto (x, g). \end{aligned} \tag{12.15}$$

Proposition 12.2. *For principal bundles, local sections/gauges are in natural 1-1 correspondence with local trivializations, according to* (12.14)–(12.15).

Proof. Verify, as an exercise, that $s_{\Phi_s} = s$ and $\Phi_{s_\Phi} = \Phi$. □

12.7 Gauge transformations

12.7.1 Global gauge group

Definition 51. A *gauge transformation* of a principal G-bundle $\pi : P \to X$ is an automorphism of P (inducing the identity map id_X on the base). The group of gauge transformations, denoted $\mathcal{G}(P)$, is called the *gauge group* of P.

By definition, a gauge transformation $F : P \to P$ is a (iso)morphism so it acts on each fibre individually. Furthermore, it respects the G-torsor structure of the fibres, i.e.,

$$F(p \cdot g) = F(p) \cdot g, \qquad \forall g \in G, p \in P_x, x \in X. \tag{12.16}$$

Remark. Conceptually, a gauge transformation preserves, at each $x \in X$, the *relationships* between the elements of P_x ("relativity of local gauges/frames"), but not the elements of P_x themselves ("no absolute gauges/frames"). In physics, the *gauge principle* says that only quantities invariant under gauge transformations are observable. We can think of this as a combination of a relativity principle, and a locality principle.

Quite generally, any fibre-preserving map $F : P \to P$ can be uniquely represented as a right-multiplication map

$$F : p \mapsto p \cdot \sigma_F(p), \tag{12.17}$$

for some function $\sigma_F : P \to G$. In view of (12.16), F is a gauge transformation iff the implementing map σ_F obeys the constraint

$$(p \cdot g) \cdot \sigma_F(p \cdot g) \equiv F(p \cdot g) = F(p) \cdot g \equiv (p \cdot \sigma_F(p)) \cdot g = (p \cdot g) \cdot (g^{-1}\sigma_F(p)g),$$

i.e.,

$$\sigma_F(p \cdot g) = g^{-1}\sigma_F(p)g. \tag{12.18}$$

Thus, F is a gauge transformation iff its $\sigma_F : P \to G$ is a G-equivariant map, with the target space G equipped with the right G-action by conjugation.

Therefore, the gauge group of P is canonically identified with

$$\begin{aligned}\mathcal{G}(P) &\cong \mathrm{Map}(P, G)^G \\ &:= \{\text{smooth } \sigma : P \to G \,:\, \sigma(p \cdot g) = g^{-1}\sigma(p)g, \;\; \forall p \in P, g \in G\}.\end{aligned}$$

Exercise 12.4. Check that $\mathrm{Map}(P,G)^G$ is a group under pointwise multiplication, and that the correspondence

$$\mathcal{G}(P) \to \mathrm{Map}(P,G)^G, \qquad F \mapsto \sigma_F$$

is a group isomorphism.

Remark. Let us work out why a conjugation appears in (12.18). Suppose F maps an initial gauge $p \in P$ to some new gauge $F(p) \in P$. Necessarily, the new gauge is related to the initial one by some $h \in G$,

$$F(p) = p \cdot h.$$

This does not mean that for the general gauge $(p \cdot g)$, the transformed gauge frame $F(p \cdot g)$ is also given by $(p \cdot g) \cdot h$. Instead, the correct formula is

$$F(p \cdot g) = F(p) \cdot g = p \cdot h \cdot g = (p \cdot g) \cdot (g^{-1} \cdot h \cdot g).$$

In other words, it is the *conjugated* $g^{-1}hg$ which implements the change of gauge from $(p \cdot g)$ to $F(p \cdot g)$. (Compare with how the matrix representing a linear transformation is conjugated, when a basis change is made.)

12.7.2 Local gauge group

Because of the G-equivariance, an element $\sigma \in \mathrm{Map}(P,G)^G$ is already fully specified on a fibre P_x when its action on one point of P_x is given. So, naïvely, we just need to pick one point p_x in each fibre P_x, and a function $f : X \to G$, to obtain an element of $\mathrm{Map}(P,G)^G$. The problem is that we cannot do this smoothly, because we cannot pick the points p_x smoothly and globally unless P is trivializable. Nevertheless, this idea is workable locally.

For the local description of gauge transformations, consider an open subset $U \subset X$, over which $P|_U$ is trivializable. The "local gauge group" of the local principal bundle $P|_U$ is

$$\mathcal{G}(P|_U) = \mathrm{Map}(P|_U, G)^G.$$

Fixing some reference local gauge $s : U \to P|_U$, we can convert $\sigma \in \mathrm{Map}(P|_U, G)^G$ into

$$\tau(\sigma) := \sigma \circ s \in \mathrm{Map}(U, G).$$

Conversely, given $f \in \mathrm{Map}(U, G)$, we construct an element $\eta(f) \in \mathrm{Map}(P|_U, G)^G$ by the formula

$$\eta(f)(s(x) \cdot g) = g^{-1} f(x) g, \qquad x \in U,\ g \in G. \tag{12.19}$$

(This formula is dictated by the G-equivariance requirement.)

So, we have local gauge dependent correspondences

$$\begin{aligned} \tau &: \mathrm{Map}(P|_U, G)^G \to \mathrm{Map}(U, G) \\ \eta &: \mathrm{Map}(U, G) \to \mathrm{Map}(P|_U, G)^G. \end{aligned} \tag{12.20}$$

We may verify that τ and η are group homomorphisms, and are inverses of each other. Thus, there is a group *isomorphism*

$$\mathrm{Map}(P|_U, G)^G \overset{\cong}{\longleftrightarrow} \mathrm{Map}(U, G). \tag{12.21}$$

In view of (12.21), it is tempting to use $\mathrm{Map}(U, G)$ as a simpler "definition" of the (local) gauge group of $P|_U$. However, we must keep in mind that a choice of reference gauge s is implicitly needed to convert $f \in \mathrm{Map}(U, G)$ to a gauge transformation (via (12.20)). Since one of the motivating ideas behind principal bundles is precisely to address the issue of non-canonicity of reference gauges, we avoid such a naïve definition.

Anyway, for non-trivializable P, the global gauge group is not even *identifiable* with $\mathrm{Map}(X, G)$ because there does not exist any global gauge at all. (Exception: see Section 12.8.)

12.7.3 Change of local gauge

After composing with a gauge transformation $F : P \to P$, a (local) gauge $s : U \to P$ gets transformed to a different (local) gauge. For example, if $f \in \mathrm{Map}(U, G)$, the equivariant map $\eta(f) : P|_U \to G$ of (12.19) is the gauge transformation which modifies s to become the local gauge

$$x \mapsto s(x) \cdot (\eta(f)(s(x))) = s(x) \cdot f(x). \tag{12.22}$$

Recall from Prop. 12.2 that a change of local gauge is equivalent to a change of local trivialization. Suppose we switch from a local trivialization

Φ_α to a local trivialization Φ_β. Over $U_\alpha \cap U_\beta$, the associated local gauges are related by

$$\begin{aligned} s_{\Phi_\beta}(x) = \Phi_\beta^{-1}(x,e) &= \Phi_\alpha^{-1} \circ \Phi_\alpha \circ \Phi_\beta^{-1}(x,e) \\ &= \Phi_\alpha^{-1}(x, g_{\alpha\beta}(x)) \qquad\qquad \text{(Eq. (12.3))} \\ &= \Phi_\alpha^{-1}(x,e) \cdot g_{\alpha\beta}(x) \\ &= s_{\Phi_\alpha}(x) \cdot g_{\alpha\beta}(x), \qquad\qquad x \in U_\alpha \cap U_\beta, \end{aligned} \tag{12.23}$$

with $g_{\alpha\beta} \in \mathrm{Map}(U_\alpha \cap U_\beta, G)$ being the transition function. Because of the resemblance between (12.23) and (12.22), the transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is sometimes also called a local gauge transformation. But beware that the simple formula (12.23) only works for converting the α gauge to the β gauge. To get to some other γ gauge, we need a different transition function $g_{\alpha\gamma}$.

12.8 Abelian gauge transformation

Example 12.6. If G is Abelian, the conjugation action is trivial, thus

$$\mathrm{Map}(P,G)^G = \{\sigma : P \to G \text{ smooth } : \sigma(p \cdot g) = \sigma(p) \quad \forall p \in P,\, g \in G\}.$$

Thus, an element σ of $\mathrm{Map}(P,G)^G$ is constant on each fibre, so it descends to a map depending only on X. Explicitly,

$$\mathrm{Map}(X,G) \xrightarrow{\cong} \mathrm{Map}(P,G)^G, \qquad f \mapsto f \circ \pi,$$

and the gauge group is just $\mathcal{G}(P) \cong \mathrm{Map}(X,G)$ (globally and canonically).

Example 12.7. Consider the trivial principal bundle $P \cong S^1 \times \mathrm{U}(1)$, with S^1 the unit circle. For each integer $n \in \mathbb{Z}$, the map $\sigma_n : e^{i\theta} \mapsto e^{in\theta}$ is an element of $\mathrm{Map}(S^1, \mathrm{U}(1))$, so it implements a gauge transformation of P. Explicitly,

$$(e^{i\theta}, e^{i\phi}) \mapsto (e^{i\theta}, e^{i(\phi + n\theta)}).$$

For distinct n, the σ_n are not smoothly deformable to each other within $\mathrm{Map}(S^1, \mathrm{U}(1))$. Actually, any $\sigma \in \mathrm{Map}(S^1, \mathrm{U}(1))$ is smoothly deformable to some σ_n, and the integer label n is called the *the winding number* of σ. Thus, the gauge group of P is partitioned into deformation classes labelled by $n \in \mathbb{Z}$. Those σ with non-zero winding number are called *large gauge transformations*, because they are not obtainable from the trivial identity transformation by a "small" deformation.

13 Fibre bundles and structure groups

In Section 2.1, we emphasized that a vector $v \in V$ is a pairing of a frame $\beta : \mathbb{K}^n \xrightarrow{\cong} V$ with numerical component vectors $\mathbf{v} \in \mathbb{K}^n$, in many equivalent ways,

$$v = \beta(\mathbf{v}) = (\beta \circ g)(g^{-1}\mathbf{v}), \qquad \forall g \in \mathrm{GL}(n). \tag{13.1}$$

Therefore, we can recover V as a set of equivalence classes,

$$V = \Big(\text{Frames} \times \mathbb{K}^n\Big)\Big/_{(\beta,\mathbf{v})\sim(\beta\cdot g, g^{-1}\mathbf{v})}. \tag{13.2}$$

Fixing a choice of β, we get representatives $(\beta, \mathbf{v})$ for the equivalence classes, then the right side of (13.2) inherits a vector space structure from $\mathbb{K}^n$. This vector space structure does not depend on the choice of β.

The construction (13.2) can be generalized to recover a vector bundle E from the principal frame bundle $\mathrm{Fr}(E)$. In fact, many important bundles in physics are "associated" to from principal bundles in this manner.

13.1 Associated fibre bundles

Let $\pi : P \to X$ be a principal G-bundle over X. Let ρ be a *left* G-action on a manifold F. This means that

$$\begin{aligned} G \times F &\to F \\ (g, \xi) &\mapsto \rho(g) \cdot \xi \end{aligned} \tag{13.3}$$

is a smooth map satisfying

$$\rho(e) \cdot \xi = \xi, \qquad \rho(g_1 g_2) \cdot \xi = \rho(g_1) \cdot \rho(g_2) \cdot \xi, \qquad \forall\, g_1, g_2 \in G,\, \xi \in F.$$

For convenience, we will just write

$$g \cdot \xi \equiv \rho(g) \cdot \xi.$$

Given such a ρ, the set $P \times F$ is equipped with the right G-action (compare (13.1)),

$$(p, \xi) \cdot g := (p \cdot g, g^{-1} \cdot \xi).$$

Pass to the equivalence classes modulo this G-action,

$$P \times_\rho F := (P \times F)\big/_{(p,\xi)\sim(p\cdot g, g^{-1}\cdot\xi)}.$$

Notice that the projection map

$$\begin{aligned} \pi_\rho : P \times_\rho F &\to X \\ [p, \xi] &\mapsto \pi(p) \end{aligned}$$

makes sense on equivalence classes, since $\pi(p \cdot g) = \pi(p)$ for all $g \in G$.

Exercise 13.1. For each $x \in X$, check that

$$\pi_\rho^{-1}(x) = \{[p_x, \xi] \,:\, \xi \in F\}$$

where p_x can be chosen to be any reference point in $P_x = \pi^{-1}(x)$.

Exercise 13.1 says that each fibre of $\pi_\rho : (P \times_\rho F) \to X$ is in *non-canonical* bijection with the *typical fibre* F. Below, we sketch how the fibres $\pi_\rho^{-1}(x)$ are smoothly assembled together.

Give $P \times_\rho F$ the quotient topology, then $\pi_\rho : P \times_\rho F \to X$ is a continuous surjection (Exercise). Let $\Phi : \pi^{-1}(U) \to U \times G$ be a local trivialization of P; equivalently, we have the local gauge

$$s_\Phi : U \to P, \qquad x \mapsto \Phi^{-1}(x, e).$$

"Attach" F-valued "components" to this local gauge,

$$\begin{aligned} \Phi^{(\rho)} : \pi_\rho^{-1}(U) &\to U \times F \\ [s_\Phi(x), \xi] &\mapsto (x, \xi), \end{aligned} \tag{13.4}$$

to get a continuous local trivialization of $\pi_\rho^{-1}(U)$.

Exercise 13.2. Let (U_α, Φ_α), (U_β, Φ_β) be two local trivializations of a principal G-bundle P, with smooth transition function $g_{\beta\alpha} : U_\beta \cap U_\alpha \to G$. Let $\Phi_\alpha^{(\rho)}, \Phi_\beta^{(\rho)}$ be the corresponding (continuous) local trivializations of $\pi_\rho : P \times_\rho F \to X$, as in (13.4). Verify the transition formula

$$\begin{aligned} \Phi_\beta^{(\rho)} \circ (\Phi_\alpha^{(\rho)})^{-1} : (U_\beta \cap U_\alpha) \times F &\to (U_\beta \cap U_\alpha) \times F \\ (x, \xi) &\mapsto \big(x, \rho(g_{\beta\alpha}(x))(\xi)\big). \end{aligned} \tag{13.5}$$

By smoothness of ρ in the sense of (13.3), the transition map $\rho(g_{\beta\alpha}(-))$ in (13.5) is smooth. So $P \times_\rho F$ is a manifold, with smooth projection map π_ρ, and admits a bundle atlas of local trivializations $\Phi_\alpha^{(\rho)}$ induced from those of P.

Definition 52. The structure $\pi_\rho : P \times_\rho F \to X$ constructed above is called the *fibre bundle associated with the principal G-bundle $\pi : P \to X$ and the G-action ρ on the typical fibre F.*

More briefly, we just say that $P \times_\rho F$ is an *associated fibre bundle*. In the special case where ρ is a linear representation of G on a vector space F, the associated fibre bundle is called an *associated vector bundle*.

Example 13.1. Recall that the frame bundle $\mathrm{Fr}(E)$ of a vector bundle E is a principal $\mathrm{GL}(n)$-bundle. Let ρ be the defining action of $\mathrm{GL}(n)$ on $\mathbb{K}^n$ by matrix multiplication. Then $\mathrm{Fr}(E) \times_\rho \mathbb{K}^n$ is an associated vector bundle with transition functions $\rho(g_{\beta\alpha}) = g_{\beta\alpha}$. Thus $\mathrm{Fr}(E) \times_\rho \mathbb{K}^n$ is canonically isomorphic to E.

This is a suitable point to introduce the general notion of (smooth) fibre bundles:

Definition 53. A *fibre bundle* with *typical fibre* manifold F on which the *structure Lie group* G acts, is a smooth surjective map $\pi : E \to X$, such that every $x \in X$ lies in an open neighbourhood U with $E|_U := \pi^{-1}(U)$ being locally trivializable: there is a diffeomorphism

$$\Phi : E|_U \to U \times F \quad \text{such that} \quad \pi_U \circ \Phi = \pi \text{ on } E|_U.$$

Furthermore, for each pair $(U_\alpha, \Phi_\alpha), (U_\beta, \Phi_\beta)$ of local trivializations, we have

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x, \xi) = (x, g_{\beta\alpha}(x) \cdot \xi), \qquad x \in U_\beta \cap U_\alpha, \xi \in F,$$

with the *transition functions* $g_{\beta\alpha} : U_{\alpha\beta} \to G$ being smooth.

For a submanifold $U \subset X$, a smooth map $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$ is called a *(local) section* of E over U. The space of sections over U is denoted $\Gamma(E|_U)$. If $U = X$, then s is called a *global section* of E.

Example 13.2. Inspecting (12.3)–(12.4), a principal G-bundle is an example of a fibre bundle with typical fibre G, and structure Lie group G acting on G itself by left multiplication. Beware that left multiplication only makes sense locally, after identifying $P|_U$ with $U \times G$. There is no global left G-action on the bundle P.

Example 13.3. A rank-n vector bundle is a fibre bundle, with structure group $\mathrm{GL}(n)$ acting on the typical fibre $\mathbb{K}^n$. In Section 13.4, we will put extra structures on vector bundles, and "reduce its structure group" to certain subgroups (or covering groups) of $\mathrm{GL}(n)$.

Remark. A fibre bundle admits local sections everywhere, but may not have any global section. A vector bundle always has the zero section as a global section. But a vector bundle may not admit any global *nowhere-vanishing* section.

13.2 Operations on vector bundles

On vector spaces E, F, we have algebraic operations,

- Duals E^*;
- Direct sums $E \oplus F$;
- (Symmetrized/antisymmetrized) tensor products $E \otimes F$;
- $\mathrm{Hom}(E, F) \cong F \otimes E^*$;
- Complexification/realification;
- Complex conjugate $\overline{E}$.

These algebraic operations generalize to operations on vector bundles E, F over X.

For example, we can take the direct sum $E \oplus F$ of a rank-m and a rank n-vector bundle. Use local trivializations $E|_U \cong U \times \mathbb{K}^m$ and $F|_U \cong U \times \mathbb{K}^n$ to construct $(E \oplus F)|_U \cong U \times (\mathbb{K}^m \oplus \mathbb{K}^m)$. The transition functions are valued in

$$\mathrm{GL}(m) \times \mathrm{GL}(n) \subset \mathrm{GL}(m+n),$$

so $E \oplus F$ is a rank-$(m+n)$ vector bundle.

We can construct the contangent bundle T^*X as an associated bundle to the the tangent frame bundle $\mathrm{Fr}(TX)$ — take $\mathrm{GL}(n)$ to act on $\mathbb{R}^n$ in the *contragredient representation* $g \mapsto (g^{-1})^{\mathrm{t}}$. Similarly, tensor product bundles can be constructed as associated vector bundles for representations of $\mathrm{GL}(n)$ on tensor product vector spaces. We will return to this when we discuss differential forms.

13.3 Vector bundle metrics and orientation

If E is a real vector bundle, then a *Euclidean bundle metric* is a section

$$\langle \cdot, \cdot \rangle_E \in \Gamma(E^* \otimes E^*)$$

which restricts to an inner product $\langle \cdot, \cdot \rangle_x$ on each E_x, $x \in X$. Then E is called a *Euclidean vector bundle*. For example, a *Riemannian metric* on X is a Euclidean bundle metric on TX. Given two sections $\psi, \tilde{\psi}$ of E, their pointwise inner products assemble into a real-valued function,

$$\langle \psi, \tilde{\psi} \rangle_E \in C^\infty(X)$$
$$\langle \psi, \tilde{\psi} \rangle_E(x) := \langle \psi(x), \tilde{\psi}(x) \rangle_x.$$

Similarly, if E is a complex vector bundle, a *Hermitian bundle metric* is a section

$$\langle \cdot, \cdot \rangle_E \in \Gamma(\overline{E}^* \otimes E^*)$$

which restricts at each $x \in X$ to a Hermitian inner product on E_x. Then E is called a *Hermitian vector bundle*.

Two bases for a real vector space V are said be *equivalently oriented* if the change-of-basis matrix has positive determinant. An *orientation* of V is a choice of equivalence class of bases. The set $\mathcal{O}_V$ of orientations for V is a $\mathbb{Z}_2$-torsor. Exercise: after learning about orientations below, think about the meaning of "orientation bundle" for a real vector bundle, as a principal $\mathbb{Z}_2$-bundle.

Let E be a real vector bundle. A local trivialization $\Phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ determines a local frame,

$$x \mapsto (s_1(x), \ldots, s_n(x)), \qquad s_i(x) = \Phi_\alpha^{-1}(x, e_i),\ x \in U_\alpha.$$

This assigns a basis, thus an orientation, to each E_x, $x \in U_\alpha$. If we use another local trivialization $\Phi_\beta : U_\alpha \to U_\alpha \times \mathbb{R}^n$, we obtain another orientation on E_x for $x \in U_\beta \cap U_\alpha$. The two orientations agree at x iff the transition function satisfies

$$\operatorname{sgn}(\det g_{\beta\alpha}(x)) = 1.$$

A bundle atlas is *oriented* if all the transition functions have positive determinant. In this case, we obtain unambiguous orientations $\mathcal{O}_{E_x}$ on E_x for all $x \in X$, by taking the basis determined by any choice of Φ_α in the oriented atlas. We say that E is *orientable* if it admits an oriented bundle atlas. An *orientation* on E is a choice of maximal oriented bundle atlas.

13.3.1 Restricted frame bundles

If E is a Euclidean vector bundle, we can restrict attention to local trivializations which preserve all the fibre metrics. Such local trivializations identify the E_x with $\mathbb{R}^n$ as an inner product space, and transition functions between them will be $\mathrm{O}(n)$-valued. The local frames determined by such local trivializations will be local *orthonormal frames*. There is an *orthonormal frame bundle* $\mathrm{Fr}^{\mathrm{O}}(E)$ inside the full frame bundle $\mathrm{Fr}(E)$, and this $\mathrm{Fr}^{\mathrm{O}}(E)$ has a principal $\mathrm{O}(n)$-bundle structure.

We recover E as a Euclidean vector bundle, by taking the associated vector bundle construction with $\mathbb{R}^n$ given the standard inner product, and $\mathrm{O}(n)$ acting in the defining representation on $\mathbb{R}^n$ (as orthogonal transformations).

If E is an oriented vector bundle, its *oriented frame bundle* $\mathrm{Fr}^{+}(E)$ is a principal $\mathrm{GL}(n,\mathbb{R})^{+}$-bundle. If E is an oriented Euclidean vector bundle, its *oriented orthonormal frame bundle* $\mathrm{Fr}^{\mathrm{SO}}(E)$ is a principal $\mathrm{SO}(n)$-bundle.

If E is a Hermitian vector bundle, its *orthonormal frame bundle* $\mathrm{Fr}^{\mathrm{U}}(E)$ is a principal $\mathrm{U}(n)$-bundle.

Remark. One often refers to various frame bundles of a (oriented, Riemannian) manifold X. These refer to the frame bundles of TX.

13.4 Gauge transformations of vector bundles with reduced structure group

The *endomorphism bundle* of a vector bundle E is $\mathrm{End}(E) := E \otimes E^*$. The terminology arises because $T \in \Gamma(\mathrm{End}(E))$ acts on $v \in \Gamma(E)$ in the obvious pointwise manner,

$$T : \Gamma(E) \to \Gamma(E), \qquad (T \cdot v)(x) = T(x)(v(x)),$$

and $T \cdot (v \cdot f) = (T \cdot v) \cdot f$, for all $f \in C^\infty(X)$. Elements of $\Gamma(\mathrm{End}(E))$ are called *bundle endomorphisms*, and they can be linearly combined, and composed. There is an identity bundle endomorphism, and the invertible bundle endomorphisms form a group, denoted $\mathrm{GL}(E)$.

If E has some extra geometric structure, the local trivializations which preserve this structure will take values in some *reduced* structure group $G \subset \mathrm{GL}(n)$. For example, $G = \mathrm{U}(n), \mathrm{O}(n), \mathrm{SO}(n)$ etc. We can restrict to those bundle endomorphisms $T \in \mathrm{GL}(E)$ for which each $T(x)$ preserves

the extra structure on E_x. Such a bundle endomorphism is called a *gauge transformation of* E as a vector bundle with structure group G. These restricted bundle endomorphisms form a subgroup, denoted $\mathcal{G}(E)$.

Let $F \in \mathcal{G}(P)$ be a gauge transformation of a principal G-bundle P. We know that F is represented as right-multiplication by an equivariant map $\sigma_F : P \to G$. On any associated vector bundle $E = P \times_\rho V$, there is a corresponding gauge transformation T_F (in the sense of the previous paragraph), as follows. For $[p, \xi] \in E$, define

$$T_F[p, \xi] := [p \cdot \sigma_F(p), \xi] \equiv [p, \rho(\sigma_F(p)) \cdot \xi], \tag{13.6}$$

One checks that (Exercise)

- The map T_F in Eq. (13.6) defines a bundle endomorphism of E as a vector bundle with structure group $\rho(G)$.
- If ρ is a faithful (i.e. injective) representation, then $F \mapsto T_F$ is a group isomorphism $\mathcal{G}(P) \to \mathcal{G}(E)$.

13.5 Reduction of structure group

Definition 54. For $i = 1, 2$, let $\phi : G_1 \to G_2$ be a Lie group homomorphism, and $\pi_i : P_i \to X$ be a principal G_i-bundle. A map $F : P_1 \to P_2$ is called a *ϕ-reduction of* P_2 if

- $\pi_2 \circ F = \pi_1$,
- $F(p \cdot g) = F(p) \cdot \phi(g), \qquad g \in G_1, p \in G_1.$

In particular, if ϕ is the embedding of a Lie subgroup $H \subset G$, we simply call P_1 a *H-reduction of* P_2.

Remark. The map F restricts to a map $P_{1,x} \to P_{2,x}$ of fibres. Picking $p \in P_{1,x}$ gives an identification $P_{1,x} \cong G_1$. Likewise $F(p) \in P_{2,x}$ gives an identification $P_{2,x} \cong G_2$. Then $F|_{P_{1,x}}$ is identified with the homomorphism $\phi : G_1 \to G_2$.

Let P be a principal G-bundle, and consider an associated vector bundle $E = P \times_\rho V$. Suppose V is equipped with a G-invariant inner product,

$$\langle \xi, \zeta \rangle_V = \big\langle \rho(g)\xi, \rho(g)\zeta \big\rangle_V, \qquad \xi, \zeta \in V,\ g \in G.$$

In other words, the representation ρ of G on the typical inner product space V is unitary/orthogonal. Then the vector bundle $E = P \times_\rho V$ has a well-defined bundle metric

$$\langle [p, \xi], [p, \zeta] \rangle_E := \langle \xi, \zeta \rangle_V, \tag{13.7}$$

making it a Euclidean/Hermitian vector bundle. However, beware that the structure group need not be the whole of $\mathrm{O}(V)$ or $\mathrm{U}(V)$. Rather, E is regarded as having structure group $\rho(G) \subset \mathrm{O}(V)$ or $\rho(G) \subset \mathrm{U}(V)$.

13.6 Scalar fields?

Suppose the structure group of a fibre bundle E is reduced to the trivial group. This means that E has a bundle atlas with only trivial transition functions. Then, any pair of local trivializations $\Phi_\alpha : E|_{U_\alpha} \to U_\alpha \times F$ and $\Phi_\beta : E|_{U_\beta} \to U_\beta \times F$ restrict to the same map on the overlap. Thus, they can be combined into a single extended local trivialization $\Phi_{\alpha\beta} : E|_{U_\alpha \cup U_\beta} \to (U_\alpha \cup U_\beta) \times F$. Continuing this process over an open cover of X, we obtain a *unique* global trivialization $\Phi : E \xrightarrow{\cong} X \times F$. So sections of E become canonically identified with F-valued functions, $\Gamma(E) = C^\infty(X; F)$.

Why is it useful to think of F-valued functions as sections of a fibre bundle with trivial structure group?

The classical differential geometry of X is encoded in its frame bundle, with structure group G being some subset of $\mathrm{GL}(n)$ depending on what extra geometric structures X has. The tangent bundle, cotangent bundle, tensor bundles, etc., are bundles associated to same principal frame bundle, but for different representations of G on real vector spaces. Quantum mechanically, we also allow projective unitary representations of G on complex vector spaces. The most famous example is the "spin representation" of $G = \mathrm{SO}(3)$ (detailed later). The sections of the resulting associated vector bundles are called *fields* in physics. They come in various types — vector, n-forms, spinor, scalar — according to the representation of the group G of "local frame changes".

The trivial representation on V corresponds to V-valued *scalar fields* (sometimes called "spin-0" fields in the case of $G = \mathrm{SO}(n)$). Notice that the adjective "scalar" is used, despite the apparently "vectorial" values. The defining characteristic of a "scalar" field is that local frame changes do not affect how the field is described. This is completely different from the situa-

tion of, e.g., tangent vector fields, involving genuine *geometric* vectors (not just an n-tuple of numbers).

In quantum mechanics, a possibly-projective representation of G lifts to a genuine representation of a central extension $\tilde{G}$,

$$1 \to \mathrm{U}(1) \to \tilde{G} \to G \to 1,$$

in which the central subgroup $\mathrm{U}(1)$ is represented as scalar multiplication on a complex inner product space V ("phase rotation"). More precisely, $z \in \mathrm{U}(1)$ is represented as multiplication-by-z^n for some integer n corresponding to the relevant electric charge. Ignoring G for now, there is a principal $\mathrm{U}(1)$-bundle, which is often trivializable, and it has a connection whose curvature (explained later) is the electromagnetic field. Quantum wavefunctions are sections of an associated line bundle (or vector bundle), for the charge-n representation of $\mathrm{U}(1)$ mentioned above. This is of course, just a theoretical model, but it is able to predict some remarkable phenomena (e.g., Aharonov–Bohm effect). The main message is that quantum wavefunctions are not scalar fields (i.e., $\mathbb{C}$-valued functions), unless the charge is zero.

In the modern theory of elementary particles and strong/weak nuclear forces, there are other "internal symmetry groups" G which have nothing to do with the classical geometry of the spatial base manifold X. There is a principal G-bundle of abstract "frames" for the "internal local degrees of freedom", which is completely distinct from the classical frame bundle of X.