14 Tangent vector fields and differential forms

14.1 Integral curves of vector fields

Recall that a tangent vector at x arises as the velocity vector of some geometric curve through x.

Definition 55. Let v be a vector field on a manifold X. An *integral curve* of v is a smooth curve $\gamma : (a, b) \to X$ such that

$$\dot{\gamma}(t) = v_{\gamma(t)}, \qquad t \in (a, b).$$

Example 14.1. Consider $\mathbb{R}^2$, and the global vector field $v = \partial_2$. The vertical lines are integral curves.

Example 14.2. With polar coordinates (r, θ) on $\mathbb{R}^2 \backslash \{(x, 0) : x \leq 0\}$, consider the vector field $v = \partial_\theta$. The curves

$$\gamma(t) = (r_0, \theta_0 + t), \qquad r_0 > 0,\ \theta_0 + t \in (-\pi, \pi)$$

are integral curves. Actually, ∂_θ extends to a vector field over $\mathbb{R}^2$, and the integral curves can be extended to $t \in \mathbb{R}$.

The problem of finding integral curves is a local one. So, we may use local coordinates to transfer the problem to some open subset $U \subset \mathbb{R}^n$. Write $v = \sum_{i=1}^n v^i \partial_i$. For a curve $\gamma : (a, b) \to U$, write

$$\gamma(t) = (\gamma^i(t), \ldots, \gamma^n(t)),$$

with $\gamma^i : U \to \mathbb{R}$ being its component functions. Then

$$\dot{\gamma}(t) = \sum_{i=1}^n \dot{\gamma}^i(t) \partial_i|_{\gamma(t)}.$$

The integral curve condition is

$$\dot{\gamma}(t) = \sum_{i=1}^n \dot{\gamma}^i(t) \partial_i|_{\gamma(t)} \overset{\text{need}}{=} \sum_{i=1}^n v^i(\gamma(t)) \partial_i|_{\gamma(t)} = v_{\gamma(t)},$$

which is a set of ODEs

$$\dot{\gamma}^i(t) = v^i(\gamma^1(t), \ldots, \gamma^n(t)), \qquad i = 1, \ldots, n, \tag{14.1}$$

for the n component functions γ^i of γ. Given initial conditions $\gamma^i(t_0) = x_0^i$, $i = 1, \ldots, n$, the ODEs (14.1) have a (unique, smooth) solution for some time interval. See Theorem D.1 of [Lee] for details of the ODE theory justifying this fact. We record this as the following proposition:

Proposition 14.1. *Let v be a vector field on a manifold X. For each point $x \in X$, there exists $\epsilon > 0$ and an integral curve $\gamma : (-\epsilon, \epsilon) \to X$ starting from x, i.e., $\gamma(0) = x$.*

Generally, integral curves might not exist for all $t \in \mathbb{R}$.

Exercise 14.1. Find an integral curve of the vector field $((x^1)^2 + (x^2)^2)\partial_1$ on $\mathbb{R}^2$, passing through the point $(1, 0)$. Check that this integral curve cannot be extended to all of $\mathbb{R}$.

Definition 56. A vector field on a manifold is *complete* if all its integral curves are defined for all $t \in \mathbb{R}$.

Here is a useful sufficient criterion for v to be complete.

Lemma 14.2 (Uniform time Lemma)**.** *Let v be a vector field on a manifold X. Suppose there exists $\delta > 0$ such that for all $x \in X$, the integral curve through $\gamma(0) = x$ is defined for at least $t \in (-\delta, \delta)$. Then v is complete.*

Proof. For each $x \in X$, let $\gamma^{(x)} : (-\delta, \delta) \to X$ be an integral curve through $x = \gamma(0)$, as hypothesized. Set $x' = \gamma^{(x)}(\frac{\delta}{2})$. By hypothesis, the integral curve $\gamma^{(x')}$ is also definable for $t \in (-\delta, \delta)$. We extend $\gamma^{(x)}$ as follows,

$$\gamma^{(x)}(t) = \begin{cases} \gamma^{(x)}(t), & t \in (-\delta, \delta), \\ \gamma^{(x')}(t - \frac{\delta}{2}), & t \in (-\frac{\delta}{2}, \frac{3\delta}{2}). \end{cases}$$

This is consistent for t in the overlapping interval $(-\frac{\delta}{2}, \delta)$, since the velocity vector of $\gamma^{(x')}$ at time $t - \frac{\delta}{2}$ equals that of $\gamma^{(x)}$ at time t. Continuing this process allows $\gamma^{(x)}$ to be defined for all $t \in \mathbb{R}$. □

14.2 Commutator of vector fields

Let us write $\mathfrak{X}(X) = \Gamma(TX)$ for the $C^\infty(X)$-module of tangent vector fields on X. Recall that each $v \in \mathfrak{X}(X)$ is a derivation which maps $C^\infty(X)$ to

$C^\infty(X)$. So it makes sense to compose $u, v \in \mathfrak{X}(X)$, but this composition is generally not a derivation (why?). Nevertheless, the *commutator*

$$[u, v] := u \circ v - v \circ u : C^\infty(X) \to C^\infty(X)$$

does define another derivation of $C^\infty(X)$ (Exercise). Therefore, we may define the *Lie bracket* operation on tangent vector fields,

$$[\cdot, \cdot] : \mathfrak{X}(X) \times \mathfrak{X}(X) \to \mathfrak{X}(X). \tag{14.2}$$

The following properties are easily verified,

$$\begin{aligned} [\lambda u + v, w] &= \lambda[u, w] + [v, w] \\ [v, w] &= -[w, v] \\ 0 &= [u, [v, w]] + [v, [w, u]] + [w, [u, v]], \qquad u, v, w \in \mathfrak{X}(X),\ \lambda \in \mathbb{R}. \end{aligned} \tag{14.3}$$

These properties say that $\mathfrak{X}(X)$ is an infinite-dimensional *Lie algebra*. However, beware that $[\cdot, \cdot]$ is $\mathbb{R}$-bilinear but not $C^\infty(X)$-bilinear:

Exercise 14.2. Verify that

$$[f \cdot u, g \cdot v] = fg \cdot [u, v] + f(u(g)) \cdot v - g(v(f)) \cdot u, \qquad u, v \in \mathfrak{X}(X), f, g \in C^\infty(X).$$

So $[\cdot, \cdot]$ is not a pointwise operation — there is no Lie bracket operation on individual tangent spaces.

Example 14.3. Over a coordinate chart U, the coordinate tangent vector fields ∂_i provide a *commuting* family of derivations of $C^\infty(U)$ — this is just the statement that partial derivatives commute.

Definition 57. Let $f : X \to Y$ be a diffeomorphism. It induces the *push-forward* map of vector fields,

$$\begin{aligned} f_* : \mathfrak{X}(X) &\to \mathfrak{X}(Y) \\ v &\mapsto f_*(v) := df \circ v \circ f^{-1}. \end{aligned} \tag{14.4}$$

The Lie bracket is a *natural* operation:

Exercise 14.3. Let $u, v \in \mathfrak{X}(X)$, and $f : X \to Y$ be a diffeomorphism. Show that

$$f_*[u, v] = [f_* u, f_* v].$$

14.3 Pullbacks of functions and 1-forms

It is customary to write $\Omega^0(X) := C^\infty(X)$, and call smooth functions "0-forms". A smooth map $f : X \to Y$ induces a *pullback* map on 0-forms,

$$\begin{aligned} f^* : \Omega^0(Y) &\to \Omega^0(X) \\ g &\mapsto g \circ f. \end{aligned}$$

Similarly, f induces a pullback of 1-forms,

$$\begin{aligned} f^* : \Omega^1(Y) &\to \Omega^1(X) \\ \omega &\mapsto f^*(\omega), \end{aligned}$$

where $f^*(\omega)$ is defined by

$$(f^*\omega)(v)(x) = \omega_{f(x)}(df_x(v_x)), \qquad v \in \mathfrak{X}(X). \tag{14.5}$$

Exercise 14.4. Let $\omega \in \Omega^1(Y)$ be a 1-form, and let $f : X \to Y$ and $g \in \Omega^0(Y)$ be smooth maps. Verify the following formulae:

- $f^*(g \cdot \omega) = (f^*(g)) \cdot f^*\omega$.
- $f^*(dg) = d(f^*(g))$.

Exercise 14.5. Let $f_1 : X \to Y$ and $f_2 : Y \to Z$ be smooth maps. Check that

$$(f_2 \circ f_1)^* = f_1^* \circ f_2^*$$

when applied to $\Omega^1(Z)$ or to $\Omega^0(Z)$.

14.4 Riemannian metrics and 2-forms

The dual space V^* of linear forms $V \to \mathbb{R}$ generalizes to the space of *bilinear forms* $V \times V \to \mathbb{R}$. A bilinear form b is *non-degenerate* if

$$b(v_1, v_2) = 0 \text{ for all } v_2 \in V \text{ implies } v_1 = 0.$$

In this case, the b induces the "musical isomorphisms"

$$\begin{aligned} \flat : V &\to V^*, \qquad v \mapsto b(-, v) \\ \sharp = (\flat)^{-1} &: V^* \to V. \end{aligned}$$

(Exercise: give a direct description of $\omega \mapsto \omega^\sharp$ in terms of the non-degenerate bilinear form b.)

A bilinear form satisfying

$$b(v_1, v_2) = \pm b(v_2, v_1).$$

is said to be *symmetric* or *antisymmetric*, according to the $\pm$ sign. The (anti)symmetric forms constitute a vector space.

Example 14.4. An inner product on V is a symmetric bilinear form g which is *positive definite* ($g(v, v) \geq 0$ with equality iff $v = 0$). It is automatically non-degenerate. So, via the isomorphism $\sharp$, there is an induced inner product on V^*,

$$g(\eta, \omega) := g(\eta^\sharp, \omega^\sharp), \qquad \eta, \omega \in V^*.$$

Example 14.5. If $\eta, \omega \in V^*$, their *tensor product* is the bilinear form

$$\eta \otimes \omega : (u, v) \mapsto \eta(u)\omega(v).$$

If $\{e_1, \ldots, e_n\}$ is a basis for V and $\{e^1, \ldots e^n\}$ is the dual basis for V^*,

$$e^i(e_j) = \delta^i_j,$$

then $\{e^i \otimes e^j\}_{i,j=1,\ldots,n}$ provides a basis for the space of bilinear forms on V (Exercise).

Example 14.6. The antisymmetric bilinear form

$$\eta \wedge \omega := \eta \otimes \omega - \omega \otimes \eta, \qquad \eta, \omega \in V^*, \tag{14.6}$$

is called the *wedge product* of η and ω. The space of antisymmetric bilinear forms is denoted $\Lambda^2(V)$, and it has dimension $\binom{n}{2}$ (Exercise).

Remark. An inner product prescribes a notion of lengths of vectors, and angles between vectors. A 2-form prescribes a "signed area" to the parallelogram in $V \times V$ spanned by an ordered pair of vectors.

The concepts of (positive-definite/symmetric/antisymmetric) bilinear forms, inner product, tensor products, extend to the setting of tangent vector fields over a manifold X. Specifically, a *bilinear form on* X can be defined as a $C^\infty(X)$-bilinear map

$$b : \mathfrak{X}(X) \times \mathfrak{X}(X) \to C^\infty(X).$$

This is a collection of bilinear forms b_x on each T_xX.

Definition 58. A positive-definite symmetric bilinear form g on X is called a *Riemannian metric*. An antisymmetric bilinear form ω on X is called a (differential) *2-form*. The space of 2-forms on X is denoted $\Omega^2(X)$.

Just as 0-forms and 1-forms can be pulled back under a smooth map $f : X \to Y$, we can also pull back a bilinear form b on Y to obtain a bilinear form f^*b on X,

$$f^*b(u, v)(x) := b_{f(x)}(df_x(u_x), df_x(v_x)), \qquad u, v \in \mathfrak{X}(X). \tag{14.7}$$

It is easy to see that Riemannian metrics get pulled back to Riemannian metrics, and 2-forms get pulled back to 2-forms. Furthermore, the wedge product is compatible with pullback,

$$f^*(\eta \wedge \omega) = f^*\eta \wedge f^*\omega, \qquad \eta, \omega \in \Omega^1(Y). \tag{14.8}$$

Example 14.7. For the inclusion map $\iota : Z \hookrightarrow X$ of a submanifold, the pullback map $\iota^* : \Omega^k(X) \to \Omega^k(Z)$ is just the restriction of the relevant forms to $\mathfrak{X}(Z)$.

14.5 Differential k-forms and exterior derivative

Bilinear forms on X generalize in the obvious way to k-linear forms, which take k vector fields as arguments, and are $C^\infty(X)$-linear in each argument. Totally antisymmetric k-linear forms are called (differential) k-forms. The space of k-forms on X is denoted $\Omega^k(X)$.

Recall from Example 11.3 that d converts 0-forms into 1-forms. This is actually just the lowest-order part of the *exterior calculus* $d : \Omega^k(X) \to \Omega^{k+1}(X)$ on forms of general degree k. Let us see how this works for 1-forms.

Definition 59. Let $\omega \in \Omega^1(X)$. Its *exterior derivative* $d\omega$ is the 2-form defined by

$$d\omega(u, v) := u(\omega(v)) - v(\omega(u)) - \omega([u, v]), \qquad u, v \in \mathfrak{X}(X). \tag{14.9}$$

Exercise 14.6. Check that $d\omega$ as defined in (14.9) is indeed a 2-form. (Exercise 14.2 would be helpful.)

Exercise 14.7. Let $\lambda \in \mathbb{R}$, $\eta, \omega \in \Omega^1(X)$, $f \in \Omega^0(X)$. Check that the exterior derivative satisfies:

- $\mathbb{R}$-linearity: $d(\eta + \lambda\omega) = d\eta + \lambda d\omega$;
- Leibniz rule: $d(f \cdot \omega) = f \cdot d\omega + df \wedge \omega$;
- Differential chain complex: $d(df) = 0$;
- Coordinate formula: If in a local coordinate chart, $\omega|_U = \sum_{i=1}^n \omega_i dx^i$ for some $\omega_i \in \Omega^0(U)$, then

$$d\left(\sum_{i=1}^n \omega_i\, dx^i\right) = \sum_{i=1}^n d\omega_i \wedge dx^i = \sum_{i,j=1}^d \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i. \tag{14.10}$$

- Naturality: If $h : Y \to X$ is smooth, then $h^*(d\omega) = d(h^*\omega)$.

Exercise 14.8. Let $n = 3$. If $v = \sum_{i=1}^3 v^i \partial_i$ is a (local) vector field, with ∂_i orthonormal (a Riemannian metric is implicit), consider the associated 1-form $v^\flat = \sum_{i=1}^3 v^i dx^i$. Use the formula (14.10) to relate $dv^\flat$ with the curl of $\mathbf{v} = (v^1, v^2, v^3)$ which you learned in multivariable calculus.

In general, the *wedge product* of a k-form η and an l-form ω may be defined as

$$\eta \wedge \omega(v_1, \ldots, v_{k+l}) = \frac{1}{k!\, l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \eta(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \omega(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}),$$

where $v_1, \ldots, v_{k+l} \in \mathfrak{X}(X)$, and S_{k+l} is the set of permutations of $(k + l)$ objects. The pullback of p-forms is defined in the same way as Eq. (14.7), and it is a straightforward exercise to check that pullback commutes with wedge product, Eq. (14.8).

The exterior derivative can be axiomatically and inductively defined on k-forms.

Definition 60. The exterior derivatives $d : \Omega^k(X) \to \Omega^{k+1}(X)$ are the unique $\mathbb{R}$-linear maps satisfying

- df is the usual derivative for $f \in \Omega^0(X)$;
- $d(\eta \wedge \omega) = d\eta \wedge \omega + (-1)^k (\eta \wedge d\omega)$ when η is a k-form;
- $d^2 = 0$.

Definition 59 is consistent with Definition 60, as verfied in Exercise 14.7. The general proof of existence and uniqueness of the exterior derivatives is omitted (see, e.g., Lee's smooth manifolds textbook). We just mention that there is a local coordinate formula for $d\omega$, $\omega \in \Omega^k(X)$, analogous to Eq. (14.10).

Exercise 14.9. Show that the exterior derivative of a 2-form $\eta \in \Omega^2(X)$ is the 3-form $d\eta$ such that

$$\begin{aligned} d\eta(u,v,w) = {} & u(\eta(v,w)) + v(\eta(w,u)) + w(\eta(u,v)) \\ & - \eta([u,v],w) - \eta([v,w]u) - \eta([w,u],v), \qquad u,v,w \in \mathfrak{X}(X). \end{aligned} \tag{14.11}$$

15 Geometry of Lie group actions

15.1 Lie algebra of left-invariant vector fields

Definition 61. A vector field v on a Lie group G is *left-invariant* if

$$(L_g)_* v = v, \qquad \forall g \in G.$$

Using naturality, Exercise 14.3, we have

$$(L_g)_*[v, w] = [(L_g)_* v, (L_g)_* w] = [v, w]$$

for left-invariant vector fields v, w on G. Thus the Lie bracket can be restricted to the left-invariant vector fields.

Definition 62. The *Lie algebra* of a Lie group G, denoted $\mathfrak{g}$, is the vector space of left-invariant vector fields on G, equipped with the *Lie bracket* $[\cdot, \cdot]$ of commutator of vector fields.

Proposition 15.1. *The Lie algebra $\mathfrak{g}$ of a Lie group G is canonically linearly isomorphic to T_eG, via evaluation at the identity element $e \in G$.*

Proof. Linearity and injectivity of the evaluation-at-e map is straightforward. For surjectivity, pick $\xi \in T_eG$. There is only one possible left-invariant vector field $v^\xi : g \mapsto v^\xi_g$ satisfying $v^\xi_e = \xi$, namely,

$$v^\xi_g = ((L_g)_* v^\xi)_g = dL_g(\xi) \in T_gG, \qquad g \in G. \tag{15.1}$$

We just need to check that this v is indeed a *smooth* vector field.

By Exercise 11.9, we need to check that when v^ζ is applied to arbitrary $f \in C^\infty(G)$, the resulting function

$$\begin{aligned} v(f) : G &\to \mathbb{R} \\ g &\mapsto v^\xi_g(f). \end{aligned}$$

is smooth. Let $\gamma : (-\delta, \delta) \to G$ be a curve through e representing ξ. Then $L_g \circ \gamma$ is a curve through g representing $dL_g(\xi) = v^\xi_g$, so

$$v^\xi_g(f) = \left.\frac{d(f \circ L_g \circ \gamma)}{dt}\right|_{t=0}. \tag{15.2}$$

The right side is the partial derivative $\frac{\partial}{\partial t}|_{t=0}$ of the smooth function

$$(-\delta, \delta) \times G \to \mathbb{R}, \qquad (t, g) \mapsto f \circ L_g \circ \gamma(s),$$

so it depends smoothly on $g \in G$. □

By Prop. 15.1, T_eG can be given a Lie bracket through its canonical identification with $\mathfrak{g}$.

Corollary 15.2. *For a Lie group G, there is a canonical isomorphism*

$$TG \cong G \times \mathfrak{g}, \tag{15.3}$$

thus G is parallelizable (trivializable tangent bundle).

Proof. Every T_gG is canonically linearly isomorphic to $\mathfrak{g} \cong T_eG$, by taking $\xi \in T_eG$ to v_g^ξ. Altogether, we have a canonical diffeomorphism (15.3), which is thus a bundle isomorphism. By picking any basis to identify $T_eG \cong \mathbb{R}^n$, we get a global trivialization $TG \cong G \times \mathbb{R}^n$. □

Example 15.1. Write $\mathfrak{gl}(n, \mathbb{R})$ for the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$. Regard $\mathrm{GL}(n, \mathbb{R})$ as an open submanifold of the vector space $\mathrm{M}_n(\mathbb{R}) = \mathbb{R}^{n^2}$. Then

$$\mathfrak{gl}(n, \mathbb{R}) = T_e\mathrm{GL}(n) \cong T_e\mathrm{M}_n(\mathbb{R}) = \mathrm{M}_n(\mathbb{R}).$$

The matrix commutator provides a Lie bracket operation[19] for $\mathrm{M}_n(\mathbb{R})$. This Lie bracket coincides with the one inherited from the vector field Lie bracket on $\mathfrak{gl}(n, \mathbb{R})$, via the identification $\mathfrak{gl}(n, \mathbb{R}) \cong \mathrm{M}_n(\mathbb{R})$ (Exercise). Similarly, the Lie bracket on $\mathfrak{gl}(n, \mathbb{C}) = \mathrm{M}_n(\mathbb{C})$ coincides with the matrix commutator.

Definition 63. A *Lie group homomorphism* $f : G \to H$ is a smooth group homomorphism, while a *Lie algebra homomorphism* $\mathfrak{g} \to \mathfrak{h}$ is a Lie bracket-preserving linear map.

Exercise 15.1. Let $f : G \to H$ be a Lie group homomorphism. Regard

$$f_* := (df)_e : T_eG \to T_eH$$

as a linear map $\mathfrak{g} \to \mathfrak{h}$ through the identification of Prop. 15.1. Show that f_* is actually a Lie algebra homomorphism.

Exercise 15.2. Let G be an abelian Lie group. Explain why its Lie algebra $\mathfrak{g}$ has trivial Lie bracket, $[\cdot, \cdot] \equiv 0$.

[19] A Lie bracket operation on a vector space V is an antisymmetric bilinear map $V \times V \to V$ satisfying the Jacobi identity,

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0, \qquad u, v, w, \in V.$$

15.2 Matrix Lie group and algebra examples

Often, an abstract Lie group G is isomorphic to some *matrix Lie group*, i.e., a closed subgroup of $\mathrm{GL}(n)$ such as $\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n)$. In this case, the Lie algebra $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{R}) \cong \mathrm{M}_n(\mathbb{R})$ or $\mathfrak{gl}(n,\mathbb{C}) \cong \mathrm{M}_n(\mathbb{C})$, so the Lie bracket on $\mathfrak{g}$ can be identified with the matrix commutator (recall Example 15.1).

Exercise 15.3. Show that

- $\mathfrak{o}(n) = \mathfrak{so}(n) \subset \mathrm{M}_n(\mathbb{R})$ is the space of antisymmetric matrices.
- $\mathfrak{u}(n) \subset \mathrm{M}_n(\mathbb{C})$ is the space of antihermitian matrices.
- $\mathfrak{su}(n) \subset \mathrm{M}_n(\mathbb{C})$ is the space of traceless antihermitian matrices.

Non-isomorphic Lie groups can have isomorphic Lie algebras. For example, the Lie algebras of $\mathrm{GL}(n,\mathbb{R})^+$ and $\mathrm{GL}(n,\mathbb{R})$ coincide; similarly, $\mathfrak{o}(n) = \mathfrak{so}(n)$. A more interesting example is $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. Take

$$\mathfrak{su}(2) = \mathrm{span}_{\mathbb{R}} \left\{ \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$\mathfrak{so}(3) = \mathrm{span}_{\mathbb{R}} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \qquad (15.4)$$

A direct calculation shows that the linear map exchanging the above bases is a Lie algebra isomorphism (Exercise).

15.3 Adjoint representation

Conjugation of G by any element $g \in G$,

$$C_g : G \to G, \qquad g' \mapsto gg'g^{-1},$$

is a group automorphism of G; in fact, C_g is a diffeomorphism, thus a *Lie group automorphism*. Therefore, C_g induces a Lie algebra automorphism (Exercise 15.1),

$$\mathrm{Ad}_g := (C_g)_* = (dC_g)_e : \mathfrak{g} \to \mathfrak{g}.$$

In particular, Ad_g is an invertible linear operator on $\mathfrak{g}$. By the chain rule,

$$\mathrm{Ad}_{g_1} \circ \mathrm{Ad}_{g_2} = \mathrm{Ad}_{g_1 g_2},$$

so the map

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}), \qquad g \mapsto \mathrm{Ad}_g$$

gives a group representation of G, called the *adjoint representation.*

Exercise 15.4. Let $v \in \mathfrak{g}$ and $g \in G$. Show that $(R_g)_* v \in \mathfrak{g}$. In other words, R_g preserves the left-invariance property. Show further that

$$(R_g)_* v = \mathrm{Ad}_{g^{-1}}(v).$$

Example 15.2. Consider $\mathrm{GL}(n) \subset \mathrm{M}_n$ with Lie algebra $\mathfrak{gl}(n) = \mathrm{M}_n$. Using the matrix entries as coordinates for G, the conjugation C_g is a linear map. So

$$\begin{aligned} \mathrm{Ad}_g \equiv (dC_g)_e : \mathfrak{gl}(n) &\to \mathfrak{gl}(n) \\ A &\mapsto gAg^{-1}, \qquad g \in \mathrm{GL}(n). \end{aligned} \tag{15.5}$$

Likewise, for a matrix Lie subgroup $G \subset \mathrm{GL}(n)$, the adjoint representation of G on $\mathfrak{g} \subset \mathrm{M}_n$ is given by conjugation, (15.5).

15.4 Exponential map $\mathfrak{g} \to G$

Proposition 15.3. *Left-invariant vector fields on Lie groups are complete.*

Proof. Let v be a left-invariant vector field on G. There is an integral curve $\gamma^{(e)} : (-\delta, \delta) \to G$ for some $\delta > 0$. By left-invariance, for any $g \in G$, the g-translated curve $L_g \circ \gamma^{(e)}$ is the integral curve starting at g, and this integral curve also defined for $t \in (-\delta, \delta)$. By the uniform time Lemma 14.2, v is complete. □

By Prop. 15.3, given $v \in \mathfrak{g}$, there is an integral curve $\gamma_v : \mathbb{R} \to G$ starting at $\gamma_v(0) = e$. For some other times s, the integral curve starting at $\gamma_v(s)$ can be obtained in two ways:

1. $t \mapsto \gamma_v(s + t)$; or
2. $t \mapsto (L_{\gamma_v(s)} \circ \gamma_v)(t) = \gamma_v(s) \cdot \gamma_v(t)$.

By uniqueness, these coincide,

$$\gamma_v(s) \cdot \gamma_v(t) = \gamma(s + t), \qquad s, t \in \mathbb{R},$$

thus $\gamma_v : (\mathbb{R}, +) \to G$ is actually a *Lie group homomorphism.*

Definition 64. A *one-parameter subgroup of a Lie group G* is a curve $\gamma : \mathbb{R} \to G$ which is also a Lie group homomorphism.

Theorem 15.4. *One-parameter subgroups of a Lie group G are in bijection with $\mathfrak{g}$.*

Proof. Starting from a left-invariant vector field $v \in \mathfrak{g}$, we had just explained that its integral curve through e is a one-parameter subgroup.

In reverse, a one-parameter subgroup $\gamma : \mathbb{R} \to G$ has an initial velocity $\dot{\gamma}(0) \in T_eG = \mathfrak{g}$, which in turn gives a left-invariant vector field,

$$v : g \mapsto dL_g(\dot{\gamma}(0)) \in T_gG.$$

We will check that γ is the integral curve of v through e, thereby establishing the claimed bijection.

The homomorphism property $\gamma(s+t) = \gamma(s)\gamma(t)$ is equivalently rewritten as $\gamma \circ L_s = L_{\gamma(s)} \circ \gamma$, and this implies that

$$d\gamma \circ dL_s = dL_{\gamma(s)} \circ d\gamma, \qquad s \in \mathbb{R}.$$

Then

$$\begin{aligned} \dot{\gamma}(s) = (d\gamma)_s \left(\frac{d}{dt}\Big|_{t=s} \right) &= d\gamma_s \left((dL_s)_0 \left(\frac{d}{dt}\Big|_{t=0} \right) \right) \\ &= (dL_{\gamma(s)})_e \left(d\gamma_0 \left(\frac{d}{dt}\Big|_{t=0} \right) \right) = (dL_{\gamma(s)})_e(\dot{\gamma}(0)), \end{aligned} \tag{15.6}$$

which is the required integral curve condition. □

Definition 65. For a Lie group G with Lie algebra $\mathfrak{g}$, the *exponential map* is defined as

$$\exp : \mathfrak{g} \to G, \qquad v \mapsto \gamma_v(1),$$

where $\gamma_v : \mathbb{R} \to G$ denotes the 1-parameter subgroup with $\dot{\gamma}_v(0) = v$.

For $t \neq 0$, γ_{tv} is just a reparametrization of γ_v with curve parameter rescaled by t. Thus

$$\gamma_v : t \mapsto \gamma_v(t) = \gamma_{tv}(1) = \exp(tv), \tag{15.7}$$

and one says that $v \in \mathfrak{g}$ *generates* the one-parameter subgroup γ_v.

Example 15.3. The *matrix exponential* is the map

$$\mathrm{M}_n(\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}), \qquad A \mapsto e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

(Convergent in operator norm, say.) For $A \in \mathrm{M}_n(\mathbb{R})$, consider the map

$$\gamma_A : \mathbb{R} \to \mathrm{GL}(n,\mathbb{R}), \qquad t \mapsto e^{tA}.$$

Using matrix entries X^{ij} as coordinates on $\mathrm{GL}(n,\mathbb{R})$, we see that γ_A is a smooth map which satisfies

$$\begin{aligned} \gamma_A(0) &= 1_n, \\ \gamma_A(s+t) = e^{(s+t)A} = e^{sA}e^{tA} &= \gamma_A(s)\gamma_A(t), \qquad s,t \in \mathbb{R}, \\ \dot{\gamma}_A(0) &= \sum_{i,j=1}^{n} A^{ij} \frac{\partial}{\partial X^{ij}}\Big|_{1_n} \in T_{1_n}\mathrm{GL}(n,\mathbb{R}). \end{aligned}$$

So, with the usual identification $T_{1_n}\mathrm{GL}(n,\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R})$, the curve γ_A is the one-parameter subgroup of $\mathrm{GL}(n,\mathbb{R})$ corresponding to $A \in \mathfrak{gl}(n,\mathbb{R})$. Thus the Lie-theoretic exponential map (Definition 65) is coincides with the matrix exponential. Similarly for the complex case.

Example 15.4. The exponential map depends on the target Lie group. For example, consider the *non-compact* matrix Lie group $(\mathbb{R}_{>0},\times) = \mathrm{GL}(1,\mathbb{R})$. The Lie algebra is $\mathfrak{gl}(1,\mathbb{R}) = \mathbb{R}$, spanned by the tangent vector $\partial_x|_1$. The (matrix) exponential map is

$$\mathfrak{gl}(1,\mathbb{R}) = \mathbb{R} \ni t \mapsto \exp(t\partial_x|_1) = e^t \in \mathrm{GL}(1,\mathbb{R}).$$

Compare with the *compact* matrix Lie group $\mathrm{U}(1) \subset \mathrm{GL}(1,\mathbb{C})$, which has Lie algebra $\mathfrak{u}(1)$ being the real 1-dimensional subspace $i\mathbb{R} \subset \mathrm{M}_1(\mathbb{C}) = \mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, on which the commutator is trivial. The exponential map is

$$\mathfrak{u}(1) = i\mathbb{R} \ni it \mapsto \exp(it) = e^{it} \in \mathrm{U}(1).$$

Clearly $\mathfrak{gl}(1,\mathbb{R}) \cong \mathfrak{u}(1)$ as abstract Lie algebras. However, $\mathrm{GL}(1,\mathbb{R})$ and $\mathrm{U}(1)$ are non-isomorphic Lie groups.

Exercise 15.5. Let $f : G \to H$ be a Lie group homomorphism, and $f_* : \mathfrak{g} \to \mathfrak{h}$ be the induced Lie group homomorphism. Show that the exponential map is natural, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{f_*} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{f} & H \end{array}.$$

Proposition 15.5. *Let $f : G \to H$ be a Lie group homomorphism. The adjoint representations of G and H are compatible with f in the sense that*

$$f_* \circ \mathrm{Ad}_g = \mathrm{Ad}_{f(g)} \circ f_*, \qquad g \in G.$$

Proof. Let $g \in G$ and $v \in \mathfrak{g} = T_eG$. From Exercise 15.5, we have

$$f(g \exp(tv)\, g^{-1}) = f(g) f(\exp(tv)) f(g)^{-1} = f(g) \exp(t f_*(v)) f(g)^{-1},$$

equivalently,

$$f \circ C_g \circ \exp(tv) = C_{f(g)} \circ \exp(t f_*(v)).$$

Differentiate at $t = 0$ to obtain

$$f_* \circ \mathrm{Ad}_g(v) = \mathrm{Ad}_{f(g)} \circ f_*(v).$$

□

15.5 Maurer–Cartan form

Generalizing the concept of $\mathbb{R}$-valued 1-forms, we write $\Omega^1(X, V)$ for the space of of 1-forms on X with values in a vector space V.

Definition 66. The Maurer–Cartan form on a Lie group G is the $\mathfrak{g}$-valued 1-form $\Theta \in \Omega^1(G, \mathfrak{g})$, defined by

$$\Theta_g(v_g) := (L_{g^{-1}})_*(v_g), \qquad v_g \in T_gG. \tag{15.8}$$

The idea behind Θ is to use the canonical left group action L to "push" each tangent space T_gG back to $T_eG = \mathfrak{g}$. This reverses the global parallelization construction of Corollary 15.2. In practice, Θ is a tool allowing tangent vectors anywhere on G to be compared. (Compare with how we compare tangent vectors on Euclidean space).

Exercise 15.6. Show that the Maurer–Cartan form Θ on a Lie group G:

- "Tautological",
$$\Theta(v) = v, \qquad \forall\, v \in \mathfrak{g} = T_eG; \tag{15.9}$$
- Is right-equivariant,
$$(R_g)^*\Theta = \mathrm{Ad}_{g^{-1}} \circ \Theta, \qquad g \in G \tag{15.10}$$

For $G = \mathrm{GL}(n)$, the Maurer–Cartan form is often written as

$$\Theta = g^{-1}dg.$$

This notation means that at $g' \in \mathrm{GL}(n, \mathbb{R})$, we have the matrix $(g')^{-1}$ multiplied into the matrix of coordinate 1-forms at g',

$$(g^{-1}dg)_{g'} = (g')^{-1} \cdot \begin{pmatrix} (dX^{11})_{g'} & \cdots & (dX^{1n})_{g'} \\ \vdots & \ddots & \vdots \\ (dX^{n1})_{g'} & \cdots & (dX^{nn})_{g'} \end{pmatrix}, \tag{15.11}$$

thus $g^{-1}dg$ is a $\mathrm{M}_n(\mathbb{R}) = \mathfrak{gl}(n, \mathbb{R})$-valued 1-form. One checks that $g^{-1}dg$ is indeed the Maurer–Cartan form of $\mathrm{GL}(n)$ according to Definition 66.

For $\mathrm{GL}(n, \mathbb{C})$, the coordinate X^{ij} is replaced by a complex coordinate $Z^{ij} = X^{ij} + iY^{ij}$. The Maurer–Cartan form has the same formula, with dX^{ij} replaced by dZ^{ij}.

For a matrix Lie group $G \subset \mathrm{GL}(n)$, the Maurer–Cartan form is still given by the same formula, but the 1-forms dX^{ij} are restricted to $T_gG \subset \mathrm{M}_n(\mathbb{R})$.

Exercise 15.7. Work out the Maurer–Cartan form explicitly for the matrix Lie groups $\mathrm{U}(1)$ and $\mathrm{SU}(2)$.

15.6 Fundamental vector fields on spaces with G-action

Let G act smoothly on another manifold P on the right (by diffeomorphisms). We denote this action by the smooth map

$$\sigma : P \times G \to P, \qquad (p, g) \mapsto R_g(p) \equiv p \cdot g.$$

Let us restrict the action to the 1-parameter subgroup $\gamma_v : t \mapsto \exp(tv)$ associated to some $v \in \mathfrak{g}$,

$$\begin{aligned} \sigma : P \times \mathbb{R} &\to P \\ (p, t) &\mapsto p \cdot \exp(tv). \end{aligned}$$

Starting at $p \in P$, we have the curve $t \mapsto p \cdot \exp(tv)$ with initial velocity denoted[20]

$$v_p^\sharp := \frac{d}{dt}\Big|_{t=0} (p \cdot \exp(tv)) \in T_p P. \tag{15.12}$$

Varying the starting point, we obtain a vector field $v^\sharp$ on P.

Definition 67. Let G act on a manifold P. The vector field $v^\sharp$ defined by Eq. (15.12) is called the *fundamental vector field* on P associated to v.

Note that by construction, the integral curves of $v^\sharp$ are obtained by acting by $\exp(tv)$ on the right.

Example 15.5. Let $P = G$, with G acting on itself by right multiplication. If $v \in \mathfrak{g}$ is a left-invariant vector field on G, then at any $g \in G$,

$$v_g^\sharp = \frac{d}{dt}\Big|_{t=0} (g \cdot \exp(tv)) = (L_g)_* \left(\frac{d}{dt}\Big|_{t=0} (\exp(tv)) \right) = (L_g)_*(v_e) = v_g.$$

So in this case, $v^\sharp = v$ itself.

Example 15.6. Let $S^2 \subset \mathbb{R}^3$ be manifold of unit-length row 3-vectors (n_x, n_y, n_z), on which the matrix Lie group SO(3) acts by right matrix-multiplication. Consider the $\mathfrak{so}(3)$ element $J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It generates the 1-parameter subgroup

$$t \mapsto \exp(tJ_z) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This subgroup effects

$$(n_x, n_y, n_z) \cdot \exp(tJ_z) = (n_x \cos t + n_y \sin t, -n_x \sin t + n_y \cos t, n_z)$$

[20]The sharp notation is standard, and should not be confused with the musical isomorphism introduced earlier.

which is a clockwise rotation by angle t about the z-axis. The fundamental vector field is

$$\begin{aligned}(J_z)^\sharp_{(n_x,n_y,n_z)} &= \frac{d}{dt}\Big|_{t=0}(n_x\cos t + n_y \sin t, -n_x \sin t + n_y \cos t, n_z)\\ &= (n_y, -n_x, 0) \in T_{(n_x,n_y,n_z)}S^2.\end{aligned}$$

15.6.1 Derivative of G-action map

For later use, we record the following calculation:

Proposition 15.6. *Let $\sigma : P \times G \to P$ denote the action of a Lie group G on a manifold P. Then*

$$d\sigma_{(p,g)}(v,\xi) = (dR_g)_p(v) + (\Theta(\xi))^\sharp_{p\cdot g}, \qquad (v,\xi) \in T_pP \oplus T_gG, \tag{15.13}$$

where Θ is the Maurer–Cartan form on G.

Proof. First consider $(v,0) \in T_pP \oplus T_gG$. Represent v by a curve γ through p. Then $(v,0)$ is represented by the curve $t \mapsto (\gamma(t), g)$ through (p,g). Then $d\sigma_{(p,g)}(v,0)$ is represented by the curve

$$t \mapsto \sigma(\gamma(t), g) = R_g \circ \gamma(t).$$

Thus

$$d\sigma_{(p,g)}(v,0) = (dR_g)_p(\dot\gamma(0)) = (dR_g)_p(v).$$

Next, consider $(0,\xi) \in T_pP \oplus T_gG$. Use the Maurer–Cartan form to map ξ to $\Theta(\xi) \in \mathfrak{g} = T_eG$. The latter is the velocity vector of the exponential curve $t \mapsto \exp(t\Theta(\xi))$. So $\xi \in T_gG$ is the velocity vector of $t \mapsto g\exp(t\Theta(\xi))$, thus $(0,\xi) \in T_pP \oplus T_gG$ is the velocity vector of $t \mapsto (p, g\exp(t\Theta(\xi)))$. Then $d\sigma_{(p,g)}(0,\xi)$ is the velocity vector of $t \mapsto p\cdot g\exp(t\Theta(\xi))$, i.e., the fundamental vector field $(\Theta(\xi))^\sharp$ at $p\cdot g$. □