16 Connections on principal bundles

16.1 Canonical vertical bundle of a principal bundle

Let $\pi : P \to X$ be a principal G-bundle. Each $p \in P$ lies in a fibre $P|_{\pi(p)}$, which is a submanifold diffeomorphic to G. So we may define the *vertical tangent subspace* at p,

$$V_p := T_p(P|_{\pi(p)}) \subset T_pP,$$

which has $\dim V_p = \dim G$. A "vertical" tangent vector in V_p is represented by a curve through p lying within the fibre $P|_{\pi(p)}$. After applying π, this vertical curve collapses to a constant curve at $\pi(p)$, so

$$V_p \subset \ker d\pi_p.$$

Because π is a submersion, $d\pi_p$ is surjective, so

$$\dim \ker d\pi_p = \dim T_pP - \dim T_{\pi(p)}X = \dim P - \dim X = \dim G.$$

Thus, by dimension counting,

$$V_p = \ker d\pi_p. \tag{16.1}$$

The *vertical bundle* of P is the following subset of TP,

$$VP := \ker d\pi = \bigsqcup_{p\in P} V_p \ \subset \bigsqcup_{p\in P} T_pP = TP,$$

with projection map $VP \to P$ inherited from $TP \to P$.

As V_p is a tangent space for a G-torsor, there is a canonical identification $\mathfrak{g} \cong V_p$ for all p. Specifically,

$$\begin{aligned} \mathfrak{g} = T_eG &\xrightarrow{\cong} T_pP_{\pi(p)} = V_p \\ \xi &\mapsto \xi_p^\sharp = \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(t\xi). \end{aligned} \tag{16.2}$$

In fact, recalling Eq. (15.13)), the map

$$\begin{aligned} P \times \mathfrak{g} &\to VP \subset TP \\ (p, \xi) &\mapsto \xi_p^\sharp \end{aligned} \tag{16.3}$$

gives a global frame for VP. So VP is a vector subbundle[21] of TP. We write the inverse of (16.3) as

$$\sharp^{-1} : VP \xrightarrow{\cong} P \times \mathfrak{g}, \tag{16.4}$$

and it gives a global trivialization of VP once a basis for $\mathfrak{g} \cong \mathbb{R}^{\dim \mathrm{G}}$ is picked. This parallelizability of VP generalizes that of the G-torsor fibres (Corollary 15.2).

16.2 Connection as horizontal bundle

Any element of T_pP can be said to be vertical, or non-vertical, depending on whether it lies in V_p. But there is no canonical notion of "horizontal", so we cannot say how much of $v \in T_pP$ is "vertical" and how much is "horizontal". To achieve this, we need the extra data of a complementary subspace H_pP (such that $T_pP = H_pP \oplus V_pP$). Doing this for every $p \in P$, we arrive at the following as a first definition of "connection":

Definition 68. A *connection* on a principal G-bundle P is a choice of G-invariant *horizontal subbundle* $HP \to P$ of $TP \to P$, i.e.,

$$\begin{aligned} HP \oplus VP &= TP, \\ (R_g)_* H_p &= H_{p\cdot g}, \qquad\qquad \forall p \in P,\ g \in G. \end{aligned} \tag{16.5}$$

The second condition ensures that the notion of "horizontal" is G-invariant. In particular, it does not pick out any preferred point on each fibre.

Once a connection is specified, the derivative maps may be written as

$$d\pi_p : T_pP = H_p \oplus V_p \to T_{\pi(p)}X, \qquad p \in P,$$

and they restrict to isomorphisms

$$d\pi_p|_{H_p} : H_p \xrightarrow{\cong} T_{\pi(p)}X$$

due to (16.1). Therefore, a connection also tells us how to "lift" a base tangent vector to a horizontal tangent vector.

[21] A subset F of a rank-n vector bundle $\pi : E \to X$ is a vector subbundle of rank $k \leq n$, if every $x \in X$ is contained in a local trivialization $\Phi : E|_U \xrightarrow{\cong} U \times \mathbb{K}^n$ such that $\Phi(E|_U \cap F) = U \times \mathbb{K}^k$. Then F is an embedded submanifold in E, and $\pi|_F : F \to X$ is itself a rank-k vector bundle.

Definition 69. Let $\pi : P \to X$ be a principal G-bundle with a connection. Given a tangent vector field $u \in \mathfrak{X}(X)$ on the base, its *horizontal lift* is the unique vector field $\tilde{u}^{\mathrm{H}} \in \Gamma(HP) \subset \Gamma(TP)$ such that the following commutes:

$$\begin{array}{ccc} P & \xrightarrow{\tilde{u}^{\mathrm{H}}} & HP \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle d\pi|_{HP}} \\ X & \xrightarrow{u} & TX \end{array}.$$

Exercise 16.1. Show that for any $u \in \mathfrak{X}(X)$, its horizontal lift $\tilde{u}^{\mathrm{H}}$ is G-invariant, i.e., $(R_g)_* \tilde{u}^{\mathrm{H}} = \tilde{u}^{\mathrm{H}}$ for all $g \in G$.

Exercise 16.2. Let $v \in \Gamma(TP)$ be G-invariant, and $\xi^\sharp$ be a fundamental vector field on P. Show that $[v, \xi^\sharp] = 0$.

16.3 Connection as a 1-form

Let P be equipped with a connection. Any vector field $v \in \Gamma(TP)$ splits into a horizontal and a vertical horizontal part,

$$v = v^{\mathrm{H}} + v^{\mathrm{V}}, \qquad v^{\mathrm{H}} \in \Gamma(HP),\ v^{\mathrm{V}} \in \Gamma(VP).$$

Using the canonical vertical trivialization (16.4), we can convert v^{V} into a section of $P \times \mathfrak{g}$, i.e., a $\mathfrak{g}$-valued function on P. So we have the following $\mathfrak{g}$-valued 1-form ω on P,

$$\omega : \mathfrak{X}(P) \xrightarrow{(\cdot)^{\mathrm{V}}} \Gamma(VP) \xrightarrow{\sharp^{-1}\circ} \Gamma(P \times \mathfrak{g}) = C^\infty(P; \mathfrak{g}), \tag{16.6}$$

whose role is to measure the vertical part of a vector field on P.

This $\omega \in \Omega^1(P, \mathfrak{g})$ has the following properties:

- ω annihilates the horizontal parts of vector fields (by definition);
- ω takes a fundamental vector field $\xi^\sharp$ (which is vertical) back to ξ;

$$\omega(\xi^\sharp) = \xi, \qquad \forall\, \xi \in \mathfrak{g}. \tag{16.7}$$

- ω is G-equivariant,

$$R_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega, \qquad \forall g \in G. \tag{16.8}$$

(Exercise: deduce this from Exercise 15.4 and the first property.)

Example 16.1. Consider the underlying manifold of G as a principal G-bundle over a point. There is no extra horizontal data to specify. Eq. (16.7) is basically the defining property of the Maurer–Cartan form, while (16.8) is the equivariance property in Exercise 15.6.

Our second definition of a connection is:

Definition 70. A connection on a principal G bundle P is a $\mathfrak{g}$-valued 1-form on P which satisfies (16.7)–(16.8).

Start from a connection ω in the sense of Definition 70. Define the horizontal subspaces of T_pP to be

$$H_p = \ker(\omega_p), \qquad p \in P, \qquad p \in P,$$

so they are annihilated by ω. These H_p assemble into a vector subbundle $HP \subset TP$, and Condition (16.7)says that VP is a complementary[22] subbundle to HP. Condition (16.8) may be shown to imply G-invariance of HP in the sense of (16.5) (Exercise). Thus we recover the first Definition 68 of a connection.

Example 16.2. A trivial principal bundle $X \times G$ possesses the *trivial connection* $\omega_{\text{triv}} = \pi_G^*\Theta$, where Θ is the Maurer–Cartan form on G. In other words, when restricted to a fibre, ω_{triv} is exactly the Maurer–Cartan form on $\{x\} \times G$. The horizontal subspaces are $H_{(x,g)} = T_xX \oplus 0$, i.e., identified with the tangent spaces of X.

Suppose P is trivializable. Using a trivialization $\Phi : P \xrightarrow{\cong} X \times G$, we may pull ω_{triv} back to a $\mathfrak{g}$-valued 1-form on P. The G-equivariance of Φ ensures that this pullback is a connection on P. *There is no intrinsic "trivial connection" on P unless a preferred trivialization is specified.*

Example 16.3. Let $P = S^1 \times \mathrm{U}(1)$. Let $d\theta, d\varphi$ be the respective coordinate 1-forms on S^1 and $\mathrm{U}(1)$ (Example 11.5). For $ik \in \mathfrak{u}(1) = i\mathbb{R}$, the corresponding fundamental vector field on P is $k\frac{\partial}{\partial\varphi}$. The Maurer–Cartan form on $\mathrm{U}(1)$ is $id\varphi$. Consider the $\mathfrak{u}(1)$-valued 1-form $\omega^{(k)} := ik\,d\theta + id\varphi$. Then $\omega^{(k)}$ is a connection on P. The horizontal subspaces are spanned by $\partial_\theta - k\partial_\varphi$.

Exercise 16.3. Let ω, ω' be two connections on a principal G bundle P. Show that $\omega - \omega'$ annihilates all vertical vectors,

$$(\omega - \omega')_p(v_p) = 0, \qquad \forall\, v_p \in V_p,\ p \in P.$$

[22]Technically, one may put a Riemannian metric on P, making each H_p the orthogonal complement to V_p; then HP is readily seen to be a vector subbundle. See Theorem 5.2.2 of Hamilton for details.

16.4 Local description of connection: gauge potentials

By picking a local trivialization, one has a reference "trivial connection" ω_{triv} locally, i.e., a local reference notion of "horizontal". This reference "horizontal" will generally differ from the "true" horizontal as defined by ω. (They coincide on vertical vectors by Exercise 16.3). The "vertical difference" between these "horizontals" is described as follows:

Definition 71. Let ω be a connection on a principal G-bundle $\pi : P \to X$. Upon picking a local section/gauge $s : U \to P$, we have a $\mathfrak{g}$-valued 1-form $s^*\omega$ over U, called the *local gauge potential* of ω (with respect to s).

Recalling (16.6), we have

$$(s^*\omega)_x(u_x) = \omega_{s(x)}(ds_x(u_x)) = \sharp^{-1}\big(ds_x(u_x)^{\mathrm{V}}\big), \qquad u_x \in T_xU.$$

This is the "vertical difference" mentioned above.

Example 16.4. On the trivial principal bundle $S^1 \times \mathrm{U}(1)$, consider the connection 1-form $\omega = ikd\theta + id\varphi$ of Example 16.3. With respect to the trivial section $s_0 : e^{i\theta} \mapsto (e^{i\theta}, 1)$, the gauge potential is $s_0^*\omega = ikd\theta$. This means that when the base coordinate is increased infinitesimally by $d\theta$, the trivial section appears to also be increasing in the fibre direction by an infinitesimal amount $ik \in \mathfrak{u}(1)$, relative to the "true" ω-horizontal.

Since gauge potentials are local gauge dependent, they are not intrinsic to ω. Nevertheless, the gauge potentials of ω relative to any two choices of local gauges should be related in a systematic way. The transformation law under a change of local gauge is given in the following theorem.

Theorem 16.1. *Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection on a principal G-bundle P. Let s_α, s_β be local sections of P defined over open subsets U_α, U_β respectively. On the overlap $U_\alpha \cap U_\beta$, the two local gauge potentials are related by*

$$s_\beta^*\omega = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \circ s_\alpha^*\omega + g_{\alpha\beta}^*\Theta, \tag{16.9}$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is the transition function (local gauge transformation) implementing $s_\beta = s_\alpha \cdot g_{\alpha\beta}$, and Θ is the Maurer–Cartan form on G.

Proof. With $\sigma : P \times G \to P$ the action map, we have $s_\beta = \sigma \circ (s_\alpha, g_{\alpha\beta})$. Its derivative is

$$(ds_\beta)_x = d\sigma_{(s_\alpha(x), g_{\alpha\beta}(x))} \circ ((ds_\alpha)_x, (dg_{\alpha\beta})_x), \qquad x \in U_\alpha \cap U_\beta. \tag{16.10}$$

The general formula for $d\sigma$, Eq. (15.13), is recalled below,

$$d\sigma_{(p,g)}(v,\xi) = (dR_g)_p(v) + \Theta(\xi)^\sharp_{p\cdot g}.$$

Substitute this into Eq. (16.10),

$$\begin{aligned}(ds_\beta)_x(-) &= (dR_{g_{\alpha\beta}(x)})_{s_\alpha(x)} \circ (ds_\alpha)_x(-) \;+\; \Theta((dg_{\alpha\beta})_x(-))^\sharp_{s_\beta(x)} \\ &= (dR_{g_{\alpha\beta}(x)})_{s_\alpha(x)} \circ (ds_\alpha)_x(-) \;+\; \big((g^*_{\alpha\beta}\Theta)_x(-)\big)^\sharp_{s_\beta(x)}. \qquad (16.11)\end{aligned}$$

For convenience, we drop the reference to x. Apply ω to the first term of Eq. (16.11),

$$\begin{aligned}\omega\left(dR_{g_{\alpha\beta}} \circ ds_\alpha(-)\right) &= (R^*_{g_{\alpha\beta}}\omega)(ds_\alpha(-)) \\ &= \mathrm{Ad}_{g^{-1}_{\alpha\beta}}\big(\omega(ds_\alpha(-))\big) \qquad \text{(Eq. (16.8))} \\ &= \mathrm{Ad}_{g^{-1}_{\alpha\beta}}(s^*_\alpha\omega(-)).\end{aligned}$$

Apply ω to the second term of Eq. (16.11),

$$\omega((g^*_{\alpha\beta}\Theta)(-))^\sharp \overset{\text{Eq. (16.7)}}{=} (g^*_{\alpha\beta}\Theta)(-)$$

In total,

$$\underbrace{\omega(d(s_\beta)(-))}_{=s^*_\beta\omega(-)} = (\mathrm{Ad}_{g^{-1}_{\alpha\beta}} \circ s^*_\alpha\omega + g^*_{\alpha\beta}\Theta)(-),$$

which is the claimed transformation law, Eq. (16.9). □

Exercise 16.4. Let $s_\gamma : U_\gamma \to P$ be a third local section. Show that on $U_\alpha \cap U_\beta \cap U_\gamma$, we can obtain $s^*_\gamma\omega$ from $s^*_\alpha\omega$, by first applying $g_{\alpha\beta}$, then applying $g_{\beta\gamma}$, according to the transformation rule (16.9).

Remark. When G is abelian, the transformation law, Eq. (16.9), simplifies to

$$s^*_\beta\omega = s^*_\alpha\omega + g^*_{\alpha\beta}\Theta. \qquad (16.12)$$

16.5 Local versus global picture of connections

A $\mathfrak{g}$-valued 1-form $\mathcal{A} \in \Omega^1(U,\mathfrak{g})$ is enough to specify a connection on the trivial principal bundle $U \times G$: set

$$\omega_{(x,g)}(v_x, \xi^\sharp_g) = \mathrm{Ad}_{g^{-1}} \circ \mathcal{A}(v_x) + \xi, \qquad v_x \in T_xU,\ \xi \in T_eG = \mathfrak{g}. \qquad (16.13)$$

The gauge potential for (16.13) with respect to the trivial gauge is precisely $\mathcal{A}$.

In general, we only have an open cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ of X, local trivializations, and transition functions $g_{\alpha\beta}$ satisfying the cocycle condition (12.6). Now suppose we have a collection of $\mathfrak{g}$-valued local 1-forms $\mathcal{A}_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$, such that

$$\mathcal{A}_\beta = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \circ \mathcal{A}_\alpha + g_{\alpha\beta}^* \Theta \quad \text{over } U_\alpha \cap U_\beta, \qquad \alpha, \beta \in \mathcal{I}.$$

Then it may be shown that there exists a *unique* connection ω on P such that $\mathcal{A}_\alpha = s_\alpha^* \omega$ for each $\alpha \in \mathcal{I}$. In other words, the $\{\mathcal{A}_\alpha\}_{\alpha\in\mathcal{I}}$ can be coherently "patched" to a globally defined connection. In physics, a connection is usually specified as such a coherent family $\{\mathcal{A}_\alpha\}_{\alpha\in\mathcal{I}}$, and the $\mathcal{A}_\alpha$ are called *gauge fields*.

Remark. Recall from Exercise 16.3 that any connection ω can be obtained from a reference connection ω_0 by adding a "horizontal" G-equivariant $\mathfrak{g}$-valued 1-form η on P. In turn, the space of such 1-forms is canonically identified with the 1-forms on X valued in the *adjoint bundle* $\mathrm{Ad}\, P = P \times_{\mathrm{Ad}} \mathfrak{g}$ (Exercise). So the space of connections is an affine space over an infinite-dimensional vector space.

16.6 Curvature of connections

The exterior derivative of a k-form generalizes to a map from V-valued k-forms to V-valued $(k+1)$-forms. Concretely, we pick a basis for V and expand a V-valued k-form into $\mathbb{R}$-valued k-forms, then apply the usual exterior derivative on each component.

Intuitively: In P, imagine moving an infinitesimally small distance horizontally along u^{H}, then horizontally along v^{H}. Now do this in the opposite order. If the horizontal distribution defined by ω is "curved", there will be a vertical mismatch (an element of $V_p \cong \mathfrak{g}$) for the these two motions. The curvature properties of ω is recorded by the following device.

Definition 72. Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection on a principal G-bundle P. Its *curvature* is the $\mathfrak{g}$-valued 2-form

$$\Omega(u, v) := d\omega(u^{\mathrm{H}}, v^{\mathrm{H}}) = -\omega[u^{\mathrm{H}}, v^{\mathrm{H}}], \qquad u, v \in \mathfrak{X}(P). \tag{16.14}$$

Note: The second equality in (16.14) is due to the general formula for exterior derivative of 2-forms, Eq. (14.9),

$$d\omega(u,v) = u(\omega(v)) - v(\omega(u)) - \omega[u,v].$$

together with the fact that ω annihilates horizontal vector fields.

Recall that a connection 1-form is G-equivariant (Eq. (16.8)). Similarly,

Proposition 16.2. *The curvature Ω of a connection ω is G-equivariant,*

$$R_g^*\Omega = \mathrm{Ad}_{g^{-1}} \circ \Omega, \qquad \forall\, g \in G.$$

Proof. Note that $(R_g)_*$ maps horizontal vectors to horizontal vectors (general property of connection). Then, for any $u, v \in \mathfrak{X}(P)$, we have

$$\begin{aligned}
(R_g^*\Omega)(u,v) &= \Omega((R_g)_*u, (R_g)_*v) \\
&= d\omega(((R_g)_*u)^{\mathrm{H}}, ((R_g)_*v)^{\mathrm{H}}) \\
&= d\omega((R_g)_*u^{\mathrm{H}}, (R_g)_*v^{\mathrm{H}}) && \text{(Eq. (16.5))} \\
&= (R_g^*(d\omega))(u^{\mathrm{H}}, v^{\mathrm{H}}) \\
&= d(R_g^*\omega)(u^{\mathrm{H}}, v^{\mathrm{H}}) \\
&= \mathrm{Ad}_{g^{-1}} \circ \Omega(u,v). && \text{(Eq. (16.8))}
\end{aligned}$$

□

16.6.1 Structure equation

Definition 73. The *wedge product* of $\eta, \omega \in \Omega^1(X, \mathfrak{g})$ is the $\mathfrak{g}$-valued 2-form defined by

$$[\eta, \omega](u,v)(x) = [\eta(u)(x), \omega(v)(x)] - [\eta(v)(x), \omega(u)(x)], \qquad u, v \in \mathfrak{X}(X),$$

where $[\cdot,\cdot]$ on the right side denotes the Lie bracket in $\mathfrak{g}$.

This is like an ordinary wedge product, but instead of pointwise multiplying $\eta(u)$ and $\omega(v)$, we take the pointwise Lie bracket. In the same way, we can take wedge products of $\mathfrak{g}$-valued k-forms and l-forms.

Theorem 16.3 (Cartan structure equation)**.** *The curvature of a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ can be expressed as*

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{16.15}$$

Proof. Exercise. It is helpful to split the calculation of $\Omega(u, v)$ into three cases: (i) u, v both horizontal, (ii) u, v both vertical, (iii) u vertical and v horizontal. □

Remark. When $P = G$, the connection is the Maurer–Cartan form, and it has no curvature since there are no horizontal directions. So the Maurer–Cartan form satisfies $d\Theta + \frac{1}{2}[\Theta, \Theta] = 0$.

16.6.2 Local description of curvature: field strength

Definition 74. Let Ω be the curvature of a connection ω on a principal G-bundle P. Let $s : U \to P$ be a local section of P. Then $s^*\Omega \in \Omega^2(U, \mathfrak{g})$ is called the *local field strength* of ω.

Theorem 16.4. *Let Ω be the curvature of a connection ω on a principal G-bundle. Let $s_\alpha : U_\alpha \to P$ and $s_\beta : U_\beta \to P$ be two local gauges. On $U_\alpha \cap U_\beta$, the local field strengths are related by*

$$s_\beta^*\Omega = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \circ s_\alpha^*\Omega, \tag{16.16}$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is the transition function/local gauge transformation, $s_\beta = s_\alpha \cdot g_{\alpha\beta}$.

Proof. Exercise. (Use (16.11) and the equivariance Prop. 16.2.) The calculations are similar to those in the proof of the global version of this theorem (see Theorem 16.5). □

16.7 Gauge-transformed connections and curvatures

Let $F : P_1 \to P_2$ be a morphism of principal G-bundles, and let $\omega \in \Omega^1(P_2, \mathfrak{g})$ be a connection on P_2. Then

$$\begin{aligned} R_g^*(F^*\omega) = (F \circ R_g)^*\omega &= (R_g \circ F)^*\omega && \text{(morphism is equivariant)} \\ &= F^*(R_g^*\omega) \\ &= F^*(\mathrm{Ad}_{g^{-1}} \circ \omega) && \text{(Eq. (16.8))} \\ &= \mathrm{Ad}_{g^{-1}} \circ (F^*\omega). \end{aligned}$$

Furthermore, if $v^\sharp$ is a fundamental vector field on P_1, then

$$\begin{aligned}(dF)_p(v_p^\sharp) &= \left.\frac{d(F(p\cdot\exp(tv)))}{dt}\right|_{t=0} \\ &= \left.\frac{d(F(p)\cdot\exp(tv))}{dt}\right|_{t=0} \qquad \text{(morphism)} \\ &= v^\sharp_{F(p)},\end{aligned}$$

so

$$(F^*\omega)_p(v_p^\sharp) = \omega_{F(p)}((dF)_p(v_p^\sharp)) = \omega_{F(p)}(v^\sharp_{F(p)}) = v, \qquad \forall\, p \in P_1.$$

These calculations verify that $F^*\omega \in \Omega^1(P_1, \mathfrak{g})$ is a connection on P_1, according to the defining properties (16.8) and (16.7).

In particular, (global) gauge transformations of P act on connections on P by pullback.

Theorem 16.5. *Let ω be a connection on a principal G-bundle P. Let $\sigma_F \in \mathrm{Map}(P,G)^G = \mathcal{G}(P)$ be a gauge transformation. Then*

$$\Omega^{F^*\omega} = F^*\Omega^\omega = \mathrm{Ad}_{\sigma_F^{-1}} \circ \Omega^\omega.$$

where Ω^ω and $\Omega^{F^*\omega}$ are the respective curvatures of ω and $F^*\omega$.

Proof. For the first equality, we use the Cartan structure equation, and commutativity of pullback with wedge product,

$$\begin{aligned}F^*\Omega^\omega &= F^*\left(d\omega + \frac{1}{2}[\omega,\omega]\right) \\ &= d(F^*\omega) + \frac{1}{2}[F^*\omega, F^*\omega] = \Omega^{F^*\omega}.\end{aligned}$$

(Omitted in lecture.) For the second equality, let $u, v \in \mathfrak{X}(P)$. In the expression

$$F^*\Omega^\omega(u,v) = \Omega^\omega\left(F_*u, F_*v\right), \tag{16.17}$$

we may discard the vertical parts of the arguments on the right. Note that $F : P \to P$ is the composition

$$p \mapsto (p, \sigma_F(p)) \overset{\sigma}{\mapsto} p\cdot\sigma_F(p),$$

where $\sigma : P \times G \to P$ is the group action map. We computed $d\sigma$ in Eq. (15.13),

$$d\sigma_{(p,g)}(v_p, \xi_g) = (dR_g)_p(v_p) + \text{vertical}, \qquad v_p \in T_pP,\ \xi_g \in T_gG.$$

Substitute this into Eq. (16.17),

$$\begin{aligned} F^*\Omega^\omega(u,v) &= \Omega^\omega\left(dR_{\sigma_F(p)}(u), dR_{\sigma_F(p)}(v)\right) \\ &= R^*_{\sigma_F}\Omega^\omega(u,v) \\ &= \mathrm{Ad}_{\sigma_F^{-1}} \circ \Omega^\omega(u,v), \qquad\qquad u, v \in \mathfrak{X}(P), \end{aligned}$$

where the last equality used the G-equivariance of Ω^ω. □

Theorem 16.5 is the global version of Theorem 16.4.

16.7.1 Abelian G

Theorem 16.5 says that when G is *abelian*, the curvature $\Omega \in \Omega^2(P, \mathfrak{g})$ is *invariant* under gauge transformations. Likewise, the local version, Theorem 16.4, says that the local field strengths $s^*\Omega$ are independent of the choice of local gauge/section s. This means that if we have a collection $\{s^*_\alpha\Omega\}_{\alpha\in\mathcal{I}}$ of local field strengths corresponding to a bundle atlas of local trivializations, we can combine all of them into a single globally-defined 2-form $\mathcal{F} \in \Omega^2(X, \mathfrak{g})$.

Furthermore, this $\mathcal{F}$ is a *closed* 2-form: in a(ny) local gauge s,

$$\mathcal{F} = s^*\Omega = s^*(d\omega) = d(s^*\omega)$$

holds, so

$$d\mathcal{F} = d(d(s^*\omega)) = 0.$$

Nevertheless, it is important to remember that there may not exist a *global* gauge s such that $\mathcal{F} = d(s^*\omega)$ holds. The closed 2-form $\mathcal{F}$ is said to be *locally exact* but not necessarily *globally exact*.

16.8 Physics notation

The physicists' notation for the local gauge potentials is

$$\mathcal{A} = s^*\omega,$$

with the reference to a local gauge s usually omitted. Because pullback is compatible with wedge product and the exterior derivative, the local form of the Cartan structure equation (16.15) is just

$$\mathcal{F} = s^*\Omega = d(s^*\omega) + \frac{1}{2}[s^*\omega, s^*\omega] = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}].$$

Indeed, it is customary in physics to take

$$\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}], \tag{16.18}$$

as the *definition* of the (local) curvature 2-form of $\mathcal{A}$.

Furthermore, physicists typically works in a coordinate chart (U, φ), and expands $\mathcal{A}$ in terms of coordinate 1-forms. So $\mathcal{A}$ (more precisely, $(\varphi^{-1})^*\mathcal{A}$) is written as

$$\mathcal{A} = \sum_{i=1}^{d} \mathcal{A}_i dx^i,$$

where each component $\mathcal{A}_i$ is a smooth $\mathfrak{g}$-valued function on $\varphi(U)$. Similarly, $\mathcal{F} = \sum_{1\le i<j\le d} \mathcal{F}_{ij} dx^i \wedge dx^j$, where (Exercise, using (16.18))

$$\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i + [\mathcal{A}_i, \mathcal{A}_j].$$

Finally, G is usually a matrix Lie group, $G \subset \mathrm{GL}(n)$, so $\Theta = g^{-1}dg$ (see Eq. (15.11)). Then the second term in the gauge transformation law, Eq. (16.9), is

$$g_{\alpha\beta}^*\Theta = g_{\alpha\beta}^{-1} \cdot \begin{pmatrix} dg_{\alpha\beta}^{11} & \cdots & dg_{\alpha\beta}^{1n} \\ \vdots & \ddots & \vdots \\ dg_{\alpha\beta}^{n1} & \cdots & dg_{\alpha\beta}^{nn} \end{pmatrix},$$

with $g_{\alpha\beta}^{ij} = X^{ij} \circ g_{\alpha\beta}$ being the ij-th matrix entry of the transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n)$. It is common practice to simply write $g = g_{\alpha\beta}$, and condense the above expression into "$g^{-1}dg$". So the transformation rules under a local gauge transformation g are:

$$\begin{aligned} \mathcal{A} &\rightsquigarrow \mathcal{A}' = g^{-1}\mathcal{A}g + g^{-1}dg \\ \mathcal{F} &\rightsquigarrow \mathcal{F}' = g^{-1}\mathcal{F}g. \end{aligned} \tag{16.19}$$

Remark. Sometimes, g is replaced by g^{-1} to get a left action of local gauge transformations, and there is a factor of $\pm i$ relating physicists' "matrix Lie algebras" and the mathematicians' Lie algebras. The matrix identity $dg = -g(dg^{-1})g$ may be used to write $-dg^{-1}g$ instead of $g^{-1}dg$.