18 Spin geometry and Dirac operators

18.1 Algebra of spin

18.1.1 Historical motivation from physics

An unusual kind of "spin" angular momentum (actually magnetic moment) appeared in *non-relativistic* quantum physics experiments in the first half of the 20th century. Despite the terminology "spin", there is no classical mechanical picture of "spinning" objects which can reproduce the strange empirical findings.

At around the same time (1928), Dirac found a clever solution to a fundamental puzzle in *special relativistic* quantum mechanics. For $\mathbb{C}$-valued "scalar" wavefunctions on $\mathbb{R}^d$, the standard Schrödinger equation has kinetic energy term being $\frac{\hbar^2}{2m}$ times the Laplacian $\Delta = -\sum_{j=1}^d \partial_j^2$. This has no chance of being compatible with special relativity, since the Schrödinger equation has only first-order time derivative. Considering relativistic energy-momentum relation is $E^2 = |\mathbf{p}|^2$ (massless case, speed of light=1), the abstract Schrödinger equation $i\hbar\frac{\partial}{\partial t} = H$ should have $H^2 \sim E^2 = |\mathbf{p}|^2 \sim -\hbar^2 \sum_j \partial_j^2 = \hbar^2\Delta$. Thus we seek a self-adjoint "first-order square root of $\hbar^2\Delta$" to serve as the "relativistic Hamiltonian" H.

At first, this seems like an impossible task: $(\sum_j \partial_j)^2$ contains mixed derivatives $\partial_j\partial_k$ which are not present in the Laplacian. The key algebraic insight of Dirac can already be seen from the case of Euclidean $\mathbb{R}^2$, where the Dirac operator is

$$\not{D} = -i \begin{pmatrix} 0 & \partial_x - i\partial_y \\ \partial_x + i\partial_y & 0 \end{pmatrix} \equiv -2i \begin{pmatrix} 0 & \partial \\ \bar{\partial} & 0 \end{pmatrix}, \tag{18.1}$$

and satisfies $\not{D}^2 = \Delta \otimes 1_2$ as desired. A 2×2 matrix e_j is attached to each orthonormal tangent direction ∂_j. By choosing e_j, e_k to anticommute, the "matrix-enriched" mixed partial derivatives will cancel,

$$e_j\partial_j e_k\partial_k + e_k\partial_k e_j\partial_j = (\underbrace{e_j e_k + e_k e_j}_{0})\partial_j\partial_k = 0.$$

It should be noted that Dirac was inspired by a matrix construction of Pauli in 1927, with the latter motivated by puzzling experimental results in the angular momentum properties of quantum electrons. We will see that the

key algebraic condition $e_j e_k = -e_k e_j$ encodes the hidden geometry of "spin rotations". The fundamental concept is *gauge theory* for *spin-geometry*, not special relativity.

18.2 Clifford algebra

Let V be a real vector space. The *tensor algebra* of V is

$$\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k} = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \dots,$$

with multiplication given by tensor product. In $\mathcal{T}(V)$, $1 \in \mathbb{R}$ is the multiplicative unit. We can also take the product of two vectors $u, v \in V$, denoted

$$uv \equiv u \otimes v \in V \otimes V \subset \mathcal{T}(V).$$

Now suppose V has an inner product, with associated quadratic form

$$q(v) := \langle v|v\rangle, \qquad v \in V.$$

Definition 80. Let V be a real inner product space, with q denoting the associated quadratic form. The *Clifford algebra* is the quotient algebra

$$Cl(V) := \mathcal{T}(V)/\mathfrak{I}_q.$$

where $\mathfrak{I}_q$ is the ideal generated by the relation

$$v^2 + q(v)1, \qquad v \in V.$$

By definition, $v^2 = -q(v)$ holds in $Cl(V)$, so we have the *anticommutation relations* in $Cl(V)$,

$$\begin{aligned} uv + vu = (u+v)^2 - u^2 - v^2 &= -q(u+v) + q(u) + q(v) \\ &= -2\langle u|v\rangle, \qquad u, v \in V. \end{aligned} \tag{18.2}$$

In particular, if $\{e_i\}_{i=1,\dots,n}$ is an orthonormal basis for V, then

$$\{e_i, e_j\} \equiv e_i e_j + e_j e_i = -2\delta_{ij}. \tag{18.3}$$

Proposition 18.1. *(Optional; this is to make constructions more intrinsic.) There is an injective map $\iota_V : V \hookrightarrow \mathcal{T}(V) \to Cl(V)$. Furthermore, $Cl(V)$ is the unique $\mathbb{R}$-algebra with the property: given a linear map $f : V \to \mathscr{A}$ into an associative $\mathbb{R}$-algebra $\mathcal{A}$ satisfying $f(v)^2 = -q(v)1_{\mathscr{A}}$ for all $v \in V$, there exists a unique extension of f to an algebra homomorphism, $\hat{f} : Cl(V, q) \to \mathscr{A}$. Diagrammatically:*

$$\begin{array}{ccc} V & \xrightarrow{\iota_V} & Cl(V) \\ & \searrow^{f} & \downarrow{\scriptstyle \exists!\, \hat{f}} \\ & & \mathscr{A} \end{array} .$$

Sketch. The map

$$F : \mathcal{T}(V) \to \mathscr{A}, \qquad v_1 \otimes \cdots \otimes v_n \mapsto f(v_1) \otimes \cdots \otimes f(v_n).$$

annihilates the ideal $\mathfrak{I}_q$, so it descends to a homomorphism

$$\hat{f} : \mathcal{T}(V)/\mathfrak{I}_q \equiv Cl(V) \to \mathscr{A}.$$

□

18.2.1 Linear basis for $Cl(V)$

Let $\{e_i\}_{i=1,\ldots,n}$ be an orthonormal basis for (V, q). Let I denote an increasing sequence of indices $1 \leq i_1 < \ldots < i_k \leq n$ with length $|I| = k$. Then we use the *multi-index* notation,

$$e_I := e_{i_1} \ldots e_{i_k} \in Cl(V, q), \qquad e_\emptyset = 1.$$

Note that the maximum length of a multi-index is n.

Proposition 18.2. *$Cl(V)$ has dimension $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, with a basis given by e_I with I running over all multi-indices. Each of $Cl^0(V)$ and $Cl^1(V)$ has dimension 2^{n-1}, except for the case $n = 0$, where $Cl(V) = Cl^0(V) = \mathbb{R}$.*

Proof. Exercise. (This is algebraic, but not that easy.) □

18.2.2 Even/odd elements

The integral "tensor degree" is ill-defined on $Cl(V)$; e.g., $v \otimes v$ and $-q(v)1$ have different degrees, but they represent the same element in $Cl(V)$. Nevertheless, the parity (odd versus even) of the degree is preserved, so we may split

$$Cl(V) = Cl^0(V) \oplus Cl^1(V)$$

into the *even* subalgebra $Cl^0(V)$, while $Cl^1(V)$ is a complementary linear subspace of odd elements.

18.3 Concrete Clifford algebras $Cl_n, \mathbb{C}l_n$

18.3.1 Real Clifford algebras

We write

$$Cl_n := Cl(\mathbb{R}^n),$$

where $\mathbb{R}^n$ has the standard inner product, with orthonormal basis $\{e_i\}_{i=1,\dots,n}$. (By Prop. 18.1, for a general n-dimensional V, we have $Cl(V) \cong Cl_n$.)

$\underline{Cl_0}$**.** We have $Cl_0 = \mathbb{R}$ as a real algebra.

$\underline{Cl_1}$**.** A linear basis for Cl_1 is $\{1, e_1\}$, where $e_1^2 = -1$. Taking e_1 to be a square root of -1, we have $Cl_1 \cong \mathbb{C} \cong \overline{\mathbb{C}}$ as $\mathbb{R}$-algebras.

$\underline{Cl_2}$**.** A linear basis for Cl_2 is $\{1, e_1, e_2, e_1e_2\}$. The elements e_1, e_2, e_1e_2 are anticommuting square roots of -1. So Cl_2 is isomorphic to the algebra of quaternions $\mathbb{H} = \mathrm{span}_{\mathbb{R}}\{1, I, J, K\}$, which we recall is defined by the multiplication rule

$$I^2 = J^2 = K^2 = IJK = -1.$$

Exercise 18.1. By considering the map

$$f : \mathbb{R}^n \to Cl^0_{n+1}, \qquad e_i \mapsto e_i e_{n+1}, \qquad i = 1, \dots, n,$$

show that there is an algebra isomorphism $Cl_n \cong Cl^0_{n+1}$.

18.3.2 Complex Clifford algebras

The *complexification* of Cl_n is

$$\mathbb{C}l_n := Cl_n \otimes_{\mathbb{R}} \mathbb{C}.$$

We can scalar multiply (on the right) by complex numbers,

$$(a \otimes \mu)\nu = a \otimes \mu\nu, \qquad \mu, \nu \in \mathbb{C}.$$

We usually just write $a \otimes \mu \equiv a\mu = \mu a$. The algebra product is extended from Cl_n to $\mathbb{C}l_n$ complex bilinearly, e.g.,

$$(a \otimes \mu)(b \otimes \nu) = ab \otimes \mu\nu.$$

$\underline{\mathbb{C}l_0}$**.** This is $\mathbb{C}$ as a $\mathbb{C}$-algebra.

$\underline{\mathbb{C}l_1}$**.** There is an isomorphism of $\mathbb{C}$-algebras,

$$\mathbb{C} \oplus \mathbb{C} \to \mathbb{C}l_1, \qquad \begin{cases} (1,0) \mapsto \frac{1+e_1 i}{2}, \\ (0,1) \mapsto \frac{1-e_1 i}{2}. \end{cases}$$

$\underline{\mathbb{C}l_2}$**.** Taking

$$e_1 \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad e_2 \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \omega^{\mathbb{C}} \leftrightarrow ie_1e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we get $\mathbb{C}l_2 = \mathrm{span}_{\mathbb{C}}\{1, e_1, e_2, e_1e_2\} \cong \mathrm{M}_2(\mathbb{C})$ as $\mathbb{C}$-algebras.

The $\mathbb{C}l_n, n > 2$, are obtained by "algebraic Bott periodicity".

Proposition 18.3. *For each $n \geq 0$, there is an isomorphism $\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes \mathbb{C}l_2$.*

Proof. In $\mathbb{C}l_n$, we have odd, anticommuting elements $e_i, i = 1, \ldots, n$, each squaring to -1. Similarly, in $\mathbb{C}l_2$, we have $\tilde{e}_1, \tilde{e}_2$. In $\mathbb{C}l_{n+2}$ we have odd, anticommuting $\varepsilon_i, i = i, \ldots, n+2$, each squaring to -1. Then, the correspondence

$$\begin{aligned} \varepsilon_i &\leftrightarrow e_i \otimes \sqrt{-1}\tilde{e}_1\tilde{e}_2, & i &= 1, \ldots, n, \\ \varepsilon_{n+j} &\leftrightarrow 1 \otimes \tilde{e}_j & j &= 1, 2, \end{aligned}$$

gives the desired identification $\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes \mathbb{C}l_2$. □

18.4 Spin Lie groups

On $\mathbb{R}^n$, the basic example of an orthogonal transformation is the reflection in some hyperplane. Such a reflection is labelled by a choice of unit vector $y \in S^{n-1}$ orthogonal to the hyperplane (there are two choices!),

$$\begin{aligned} O_y = O_{-y} : \mathbb{R}^n &\to \mathbb{R}^n \\ v &\mapsto v - 2\langle y|v\rangle y. \end{aligned}$$

Theorem 18.4 (Cartan–Dieudonné)**.** *Any element of* $\mathrm{O}(n)$ *is the product of at most* n *reflections. (Proof omitted.)*

Theorem 18.4 implies that every $\mathrm{SO}(n)$ element is a product of an *even* number of reflections, $O_{y_1} \dots O_{y_{2k}}$. The spin group can be thought of as a "doubled" version of $\mathrm{SO}(n)$ which lifts the two-fold degeneracy in the labeling of reflections, $O_y = O_{-y}$. The Clifford algebra formalism provides a concrete realization of this idea.

Definition 81. The *spin group* is the group in Cl_n^0 generated by even products of unit vectors,

$$\mathrm{Spin}(n) := \{y_1 \cdots y_{2k} \,:\, k \in \mathbb{N},\ y_i \in S^{n-1} \subset \mathbb{R}^n\}.$$

Note the important fact that $y^2 = -1$ lies in $\mathrm{Spin}(n)$ for all $y \in S^{n-1}$.

Proposition 18.5. *For* $n \geq 1$*, there is a short exact sequence of Lie groups*

$$1 \to \{\pm 1\} \to \mathrm{Spin}(n) \xrightarrow{\chi} \mathrm{SO}(n) \to 1.$$

Here, for $g \in \mathrm{Spin}(n)$*,* $\chi(g)$ *denotes the orthogonal transformation*

$$\chi(g) : \mathbb{R}^n \to \mathbb{R}^n, \qquad v \mapsto gvg^{-1}. \tag{18.4}$$

Proof. In the Clifford algebra, the anticommutation relation (18.2) implies that

$$yvy = (-vy - 2\langle y|v\rangle)y = v - 2\langle y|v\rangle y = O_y(v). \tag{18.5}$$

Suppose $y_1, y_2 \in S^{n-1}$, so $y_2 y_1 = (-y_2)(-y_1) = y_2^{-1} y_1^{-1} = (y_1 y_2)^{-1}$. By (18.5),

$$(y_1y_2)v(y_1y_2)^{-1} = y_1y_2vy_2y_1 = y_1(O_{y_2}(v))y_1 = O_{y_1}O_{y_2}(v), \qquad \forall\, v \in \mathbb{R}^n,\ y_1, y_2 \in S^{n-1}.$$

Let $g = (y_1 y_2) \dots (y_{2k-1} y_{2k}) \in \mathrm{Spin}(n)$ (with $y_i \in S^{n-1}$). The last calculation implies that

$$\chi(g)(v) = gvg^{-1} = O_{y_1} \dots O_{y_{2k}}(v), \qquad v \in \mathbb{R}^n.$$

Then the Cartan–Dieudonné result shows that $\chi(g)$ is indeed a special orthogonal transformation, and also that χ is a surjective group homomorphism.

(Computation of $\ker \chi$; omitted.) Now let $g \in \ker \chi$, so

$$gv = vg, \qquad \forall\, v \in \mathbb{R}^n. \tag{18.6}$$

Set $v = e_1$ above. As g lies in the even subalgebra Cl_n^0, we may write

$$g = \sum_{|I| \text{ even},\, i_k \neq 1} g^I e_I + \sum_{|J| \text{ odd},\, j_k \neq 1} g^J e_1 e_J =: g_0 + e_1 g_1,$$

where neither g_0 nor g_1 has e_1 factors. So e_1 commutes with g_0 and anticommutes with e_1, and it follows that

$$e_1 g_0 - g_1 = e_1 g \overset{(18.6)}{=} g e_1 = g_0 e_1 + e_1 g_1 e_1 = e_1 g_0 + g_1.$$

We conclude that $g_1 = 0$, so $g = g_0$ has no e_1 factor. The same argument with $v = e_2, \dots, e_n$ shows that g does not contain any $e_2, \dots, e_{n-1}, e_n$ factor, so g can only be a scalar in Cl_n. Together with

$$g^2 = (y_1 \cdots y_{2k})^2 = \pm y_1^2 \dots y_{2k}^2 = \pm 1,$$

we conclude that $g \in \ker \chi$ has to be either $+1$ or -1. Clearly $-1 = e_1^2$ and $\chi(-1) = 1_n$, so $-1 \in \ker \chi$. Thus $\ker \chi = \{\pm 1\}$.

(Lie group property; omitted.) Let $Cl_n^\times$ denote the group of invertible elements of Cl_n; it may be checked to be an open submanifold, thus a Lie group (this is a general feature of finite-dimensional algebras regarded as vector space manifolds). The adjoint representation,

$$\mathrm{Ad} : Cl_n^\times \to \mathrm{GL}(Cl_n), \qquad \mathrm{Ad}(a) : b \mapsto aba^{-1},$$

is a continuous map. We can regard χ as the restriction of Ad to the subgroup $\mathrm{Spin}(n)$; here, $\chi(g) \in \mathrm{SO}(n)$ is promoted to an automorphism of Cl_n. Since $\mathrm{SO}(n)$ is closed, the preimage $\mathrm{Spin}(n)$ is a closed subgroup of $Cl_n^\times$, thus a Lie group (by a general closed subgroup theorem in Lie theory). $\square$

Proposition 18.6. *For $n \geq 2$, the spin group* $\mathrm{Spin}(n)$ *is connected.*

Proof. Write $g \in \mathrm{Spin}(n)$ as $g = y_1, \dots y_{2k}$ with $y_i \in S^{n-1}$. Let $\gamma_i(t)$ be a path in S^{n-1} connecting y_i to y_1, so $\Pi_{i=1}^{2k}\gamma_i(t)$ is a path in $\mathrm{Spin}(n)$ connecting g to $y_1^{2k} = (-1)^k$. This shows that g is connected to either $+1$ or -1 in $\mathrm{Spin}(n)$.

It remains to check that $+1$ and -1 are connected in $\mathrm{Spin}(n)$. For orthonormal e_1, e_2, the path

$$\gamma(t) = -\cos(t) - \sin(t)e_1e_2, \qquad t \in [0, \pi].$$

joins -1 to $+1$ in Cl_n. But we can factorize

$$\gamma(t) = \left(\cos(t/2)e_1 + \sin(t/2)e_2\right)\left(\cos(t/2)e_1 - \sin(t/2)e_2\right),$$

into the product of two paths of *unit* vectors, thus γ is a path in $\mathrm{Spin}(n)$. □

Remark 5. For $n \geq 3$, it may be shown that $\mathrm{SO}(n)$ has fundamental group $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$.

18.5 Spin Lie algebra

Recall that $\mathfrak{so}(n)$ is the space of antisymmetric $n \times n$ matrices. A natural basis consists of the matrices $E_{ij}, 1 \leq i < j \leq n$, with entries

$$\begin{cases} 1, & \text{column } i, \text{ row } j, \\ -1, & \text{column } j, \text{ row } i, \\ 0, & \text{otherwise.} \end{cases}$$

For example,

$$E_{12} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Next, since $\mathrm{Spin}(n)$ is a Lie group, it has a Lie algebra $\mathfrak{spin}(n)$. The concrete realization of $\mathrm{Spin}(n)$ as a subgroup of invertible elements of Cl_n, leads to a concrete description of $\mathfrak{spin}(n)$ as well. Here are the details.

- $Cl_n^\times$ is open in the vector space Cl_n (exercise), so it is a Lie group. Its Lie algebra $\mathfrak{cl}_n^\times$ is identified with Cl_n as a vector space.
- Note that $Cl_n \subset \mathrm{End}(Cl_n)$ via left-multiplication. So $Cl_n^\times \subset \mathrm{GL}(Cl_n)$ is a matrix Lie group, and the Lie bracket on $\mathfrak{cl}_n^\times = Cl_n$ coincides with the commutator on $\mathrm{End}(Cl_n)$ (restricted to Cl_n).
- By a further restriction, $\mathrm{Spin}(n) \subset Cl_n^\times$ has $\mathfrak{spin}(n) \subset \mathfrak{cl}_n^\times$ being a Lie subalgebra of Cl_n with commutator as Lie bracket.

An explicit linear basis for $\mathfrak{spin}(n)$ can be given as follows. For $1 \le i < j \le n$, the curve

$$\gamma_{ij} : t \mapsto \cos(t) + \sin(t) e_i e_j, \qquad 1 \le i < j \le n, \tag{18.7}$$

which lies in $\mathrm{Spin}(n)$ by the proof of Prop. 18.6. It has initial velocity

$$\dot{\gamma_{ij}}(0) = e_i e_j \in T_e \mathrm{Spin}(n) = \mathfrak{spin}(n).$$

By dimension counting, we deduce that

$$\mathfrak{spin}(n) = \mathrm{span}_{\mathbb{R}}\{e_i e_j \,:\, 1 \le i < j \le n\}.$$

Proposition 18.7. *The double-covering homomorphism $\chi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ induces the Lie algebra isomorphism*

$$\begin{aligned} \chi_* : \mathfrak{spin}(n) &\to \mathfrak{so}(n) \\ e_i e_j &\mapsto 2E_{ij}, \qquad 1 \le i < j \le n. \end{aligned}$$

Proof. Recalling the curves γ_{ij} of (18.7), the induced χ_* is given by

$$\begin{aligned} \chi_* : \mathfrak{spin}(n) &\to \mathfrak{so}(n) \\ e_i e_j &\mapsto \frac{d(\chi \circ \gamma_{ij})}{dt}\Big|_{t=0}. \end{aligned}$$

To see which antisymmetric matrix the right side is, we apply it to an arbitrary $v \in \mathbb{R}^n$. For convenience we just write γ for γ_{ij}. A preliminary observation is

$$\frac{d(\gamma(t)^{-1})}{dt}\Big|_{t=0} = -\frac{d\gamma(t)}{dt}\Big|_{t=0} = -e_i e_j.$$

Now compute

$$\begin{aligned}
\chi_*(e_ie_j)(v) &= \frac{d\big(\chi(\gamma(t))(v)\big)}{dt}\Big|_{t=0} \\
&= \frac{d}{dt}\Big|_{t=0}\big(\gamma(t)v\gamma(t)^{-1}\big) \\
&= \dot\gamma(0)v\gamma(0)^{-1} + \gamma(0)v(\dot{\gamma^{-1}})(0) \\
&= e_ie_jv - ve_ie_j \\
&= e_i(-ve_j - 2\langle e_j|v\rangle) + (e_iv + 2\langle e_i|v\rangle)e_j \qquad \text{(Eq. (18.2))} \\
&= -2e_i\langle e_j|v\rangle + 2e_j\langle e_i|v\rangle \\
&= 2E_{ij}(v),
\end{aligned}$$

which is the claimed formula for χ_*. As it is surjective, by dimension counting, χ_* is an isomorphism. □

18.5.1 Explicit Spin groups in low dimensions

Spin(1). This is the two element group $\{\pm 1\}$.

Spin(2). It is not hard to deduce that

$$\mathrm{Spin}(2) = \{a + b\,e_1e_2 \,:\, a^2 + b^2 = 1\},$$

with group inversion given by $(a, b) \mapsto (a, -b)$.

Spin(3). It is well-known that there are (non-canonical!) isomorphisms

$$\mathrm{Spin}(3) \overset{(18.9)}{\cong} \mathrm{Sp}(1) \overset{(18.11)}{\cong} \mathrm{SU}(2) = \left\{ \begin{pmatrix} w & -\bar z \\ z & \bar w \end{pmatrix} \,:\, |w|^2 + |z|^2 = 1 \right\}. \quad (18.8)$$

(Detailed constructions, omitted in lectures.) By a slight modification of Exercise 18.1, Cl_3^0 can be identified with the Clifford algebra Cl_2 by setting

$$\tilde e_1 = e_2e_3, \qquad \tilde e_2 = e_3e_1.$$

In other words,

$$Cl_3^0 \cong Cl_2 = \mathrm{span}_{\mathbb{R}}\{1, \tilde e_1, \tilde e_2, \tilde e_1\tilde e_2\} = \mathrm{span}_{\mathbb{R}}\{\underbrace{e_2e_3}_{I}, \underbrace{e_3e_1}_{J}, \underbrace{e_1e_2}_{K}\} \cong \mathbb{H}. \quad (18.9)$$

Regard $\mathbb{H}^1 = \mathbb{H}$ as the 1-dimensional "quaternionic vector space", equipped with right "scalar multiplication" by $\mathbb{H}$. The conjugate of $q = a + bI + cJ + dK \in \mathbb{H}^1$ is defined as $\overline{q} = a - bI - cJ - dK$. The "inner product" on $\mathbb{H}^1$ is $(q_1, q_2) = \overline{q_1} q_2$, so the "norm" is

$$||q||^2 = a^2 + b^2 + c^2 + d^2.$$

Left-multiplication on $\mathbb{H}^1$ by $\mathbb{H}$ is a quaternion-linear map. The group in $\mathbb{H}$ which preserves the inner product is called the *symplectic group* $\mathrm{Sp}(1)$. It is easily checked that $\mathrm{Sp}(1)$ comprises the unit-norm quaternions, thus $\mathrm{Sp}(1) \cong S^3$ is a Lie group.

Recall that $\mathrm{Spin}(3) \subset Cl_3^0$. Let us check that the algebra isomorphism $Cl_3^0 \cong \mathbb{H}$ of (18.9) restricts to a Lie group isomorphism $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$. By definition, $\mathrm{Spin}(3)$ elements are even Clifford-products $y_1 \cdots y_{2k}$, with y_i being unit 3-vectors. So

$$\mathrm{Spin}(3) \ni y_1 \cdots y_{2k} = -(y_1 e_3)(e_3 y_2) \cdots (y_{2k-1} e_3)(e_3 y_{2k}). \tag{18.10}$$

It is straightforward to check that $y_i e_3, e_3 y_i \in Cl_3^0 \cong \mathbb{H}$ are identified with unit quaternions (i.e., $\mathrm{Sp}(1)$ elements). Then (18.10) shows that $\mathrm{Spin}(3)$ elements are identified with $\mathrm{Sp}(1)$ elements.

We have established an injective homomorphism $\mathrm{Spin}(3) \hookrightarrow \mathrm{Sp}(1)$ of Lie groups. Since $\mathrm{Sp}(1) \cong S^3$ is a connected Lie group of the same dimension as $\mathrm{Spin}(3)$, general Lie group theory (omitted) shows that $\mathrm{Spin}(3) \hookrightarrow \mathrm{Sp}(1)$ is in fact an isomorphism.

There is a further accidental isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. Represent the $\mathbb{H}$ as an $\mathbb{R}$-subalgebra of $\mathrm{M}_2(\mathbb{C})$, via

$$\mathbb{H} \ni a + bI + cJ + dK = \begin{pmatrix} a - id & -c - ib \\ c - ib & a + id \end{pmatrix}. \tag{18.11}$$

It is easy to check that the matrix on the right side belongs to $\mathrm{SU}(2)$ iff $a^2 + b^2 + c^2 + d^2 = 1$ iff the left side is a unit quaternion. Thus we have the isomorphisms (18.8).

18.6 Spin angular momentum

In $\mathbb{R}^3$, recall the quantum mechanical orbital angular momentum operators L_1, L_2, L_3 of Example 10.4. A better labelling would be L_{23}, L_{31}, L_{12}. They *formally* satisfy the *angular momentum commutation relations*

$$[L_{23}, L_{31}] = i\hbar L_{12}, \tag{18.12}$$

and also for the cyclic permutation of the indices. In $\mathfrak{so}(3)$, we likewise have the Lie bracket

$$[E_{23}, E_{31}] = E_{12}, \tag{18.13}$$

so actually,

$$E_{jk} \mapsto \frac{1}{ih} L_{jk} \tag{18.14}$$

gives a representation of $\mathfrak{so}(3)$ as formally skew-adjoint operators on $L^2(\mathbb{R}^3)$.

Spin orbital momentum has a related, but rather different physical and mathematical nature. Quite generally, an arbitrary oriented Riemannian X automatically has *gauge symmetry* under the $\mathrm{SO}(d)$ rotations of oriented orthonormal frames. Each tangent vector space carries the fundamental representation of $\mathrm{SO}(d)$, thus also of $\mathfrak{so}(d)$, given by the basic real antisymmetric matrices E_{jk}. For $d = 3$, the complexification of this representation is the *spin-1 representation* or *vector representation* of $\mathfrak{so}(3) \cong \mathfrak{spin}(3)$.

What is more interesting is the representation

$$\mathfrak{so}(3) \cong \mathfrak{su}(2) = \mathrm{span}_{\mathbb{R}} \left\{ \frac{1}{2i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2i} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \frac{1}{2i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{18.15}$$

on $\mathbb{C}^2$, already encountered in (15.4). Under the isomorphism $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$, thus $\mathfrak{su}(2) \cong \mathfrak{spin}(3)$, the above matrices are actually defining a representation of

$$\mathfrak{spin}(3) = \mathrm{span}_{\mathbb{R}} \left\{ \frac{e_2 e_3}{2}, \frac{e_3 e_1}{2}, \frac{e_1 e_2}{2} \right\} \tag{18.16}$$

on $\mathbb{C}^2$, called the (complex) *spin-$\frac{1}{2}$* or *spinor representation*. Just like (18.13), we have

$$\left[\frac{e_2 e_3}{2}, \frac{e_3 e_1}{2} \right] = \frac{e_1 e_2}{2}$$

Indeed, the Hermitian matrices

$$\frac{\hbar}{2}\sigma_1 = \Sigma_{23} = \frac{i\hbar}{2} e_2 e_3, \qquad \frac{\hbar}{2}\sigma_2 = \Sigma_{31} = \frac{i\hbar}{2} e_3 e_1, \qquad \frac{\hbar}{2}\sigma_3 = \Sigma_{12} = \frac{i\hbar}{2} e_1 e_2,$$

were introduced by Pauli in 1927 as the *spin angular momentum* matrices, and they satisfy the same angular momentum commutation relations (18.12) (thus the name). Here, the factor of $\frac{1}{2}$ is needed, and it corresponds to $\chi_* : e_j e_k \mapsto 2E_{jk}$ found in Prop. 18.7.

The eigenvalues of Σ_{jk} are $\pm\frac{\hbar}{2}$, consistent with the abstract half-quantization result found in Assignment 1. Although it is often said that angular momentum is quantized to $\frac{\hbar}{2}$ units in quantum mechanics, this is theoretically inferred, not directly measured. In actual experiments, a charge $Q = qe$ particle of mass m couples to a background magnetic field, in the sense that an extra energy term

$$-\frac{Q\hbar}{2m}\sigma \cdot \mathbf{B}$$

appears in the 2×2 matrix Hamiltonian (acting on the Hilbert space of 2-component spinors). This term can be analyzed geometrically in terms of Dirac operators (defined later), but the key point to emphasize is that spin-charge-mass are concepts that should not be separately discussed (at least for physics).

18.7 Spin representations

The spinor representation (18.15)–(18.16) actually comes from a representation of $\mathbb{C}l_3 \supset Cl_2^0 \subset \mathrm{Spin}(3)$, if we take

$$e_1 \sim \pm\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad e_2 \sim \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 \sim \pm\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

This fact is important, since (following Dirac) we wanted a way to implement $e_j e_k = -e_k e_j$.

For general n, the systematic way to define the spin representations is as follows. Proposition 18.3, the initial cases $\mathbb{C}l_0, \mathbb{C}l_1$, and $\mathbb{C}l_2 \cong \mathrm{M}_2(\mathbb{C})$ imply that

$$\mathbb{C}l_n \cong \begin{cases} \mathrm{M}_{2^m}(\mathbb{C}), & n = 2m \text{ even}, \\ \mathrm{M}_{2^m}(\mathbb{C}) \oplus \mathrm{M}_{2^m}(\mathbb{C}), & n = 2m+1 \text{ odd}. \end{cases}$$

Even $n = 2m$. $\mathbb{C}l_{2m} \cong M_{2^m}(\mathbb{C})$ has only one irreducible representation up to equivalence, namely, the defining one on $\mathbb{C}^{2^m}$.

Odd $n = 2m+1$. There are two irreducible representations of $\mathbb{C}l_{2m+1} \cong \mathrm{M}_{2^m}(\mathbb{C}) \oplus \mathrm{M}_{2^m}(\mathbb{C})$, depending on which copy of $\mathrm{M}_{2^m}(\mathbb{C})$ is being non-trivially represented. (Exercise: The chirality element $\omega_{2m+1}^{\mathbb{C}}$ acts as $+1$ or -1 on each copy of $\mathrm{M}_{2^m}(\mathbb{C})$.)

We shall write S (or $S^{\pm}$ in the odd n case) for the irreducible representation space for $\mathbb{C}l_n$, and denote this Clifford algebra representation by

$$c : \mathbb{C}l_n \to \mathrm{End}(S).$$

The restriction of c to $\mathrm{Spin}(d)$ is called the *spin representation*, and is denoted

$$\delta_n : \mathrm{Spin}(n) \to \mathrm{U}(S).$$

Here, the unitarity of $\delta_n(g), g \in \mathrm{Spin}(n)$ should be checked.

Proposition 18.8. *The spin representation is compatible with Clifford multiplication, in the sense that*

$$\delta_n(g)c(v)\delta_n(g^{-1}) = c\big(\chi(g)(v)\big), \qquad \forall\, g \in \mathrm{Spin}(n),\ v \in \mathbb{R}^n. \tag{18.17}$$

Proof. Since $\delta_n(\cdot)$ is just a restriction of $c(\cdot)$, we have

$$\delta_n(g)c(v)\delta_n(g^{-1}) = c(g)c(v)c(g^{-1}) = c(gvg^{-1}) \overset{(18.4)}{=} c\big(\chi(g)(v)\big).$$

□

18.8 Spin manifold and spin structures

Definition 82. A *spin manifold* is an oriented Riemannian manifold X, equipped with a *spin structure*, i.e., a χ-reduction, $F : \mathrm{Fr}^{\mathrm{Spin}}(X) \to \mathrm{Fr}^{\mathrm{SO}}(X)$ (see Definition 54).

Example 18.1. In Assignment 3, we saw an explicit realization of a Spin(2) principal bundle which χ-reduces the oriented orthonormal frame bundle of S^2.

Remark. If $\mathbf{e} : U \to \mathrm{Fr}^{\mathrm{SO}}(X)$ is a local oriented orthonormal tangent frame field over a contractible U, then there are two choices of local spin frame fields $\tilde{\mathbf{e}}$ over U lifting $\mathbf{e}$. The choices are related by the action of $-1 \in \ker\chi \subset \mathrm{Spin}(n)$.

The existence/uniqueness of spin structures may be topologically obstructed (independent of Riemannian metric).

Example 18.2. On S^1, we have $\mathrm{Fr}^{\mathrm{SO}}(S^1) = S^1 \times \{1\}$. There are two inequivalent spin structures: $\mathrm{Fr}^{\mathrm{Spin}}(S^1)$ is either the disconnected double cover, $S^1 \times \mathbb{Z}_2$, or a connected double cover of S^1. These two spin structures are respectively called *Neveu–Schwarz* and *Ramond*.

18.9 Spin connection

Definition 83. If E has a bundle metric, then a covariant derivative ∇ on E is *metric-compatible* if

$$v(\langle\psi,\varphi\rangle_E) = \langle\nabla_v\psi,\varphi\rangle_E + \langle\psi,\nabla_v\varphi\rangle_E, \qquad \psi,\varphi\in\Gamma(E),\ v\in\mathfrak{X}(X). \tag{18.18}$$

Let X be a Riemannian manifold. By a fundamental result in Riemannian geometry, there is a unique metric-compatible covariant derivative on TX, which is additionally *torison-free*,

$$\nabla_u v - \nabla_v u - [u,v] = 0, \qquad u,v\in\mathfrak{X}(X). \tag{18.19}$$

This is called the *Levi–Civita*[24] covariant derivative ∇^{LC}.

It will be convenient to adopt the principal bundle perspective: ∇^{LC} is the induced covariant derivative on TX associated to a certain distinguished *Levi–Civita connection* ω^{LC} on the orthonormal frame bundle $\mathrm{Fr}^{\mathrm{O}}(X)$.

Definition 84. Let X be a spin manifold. The (Levi–Civita) *spin connection* on $\mathrm{Fr}^{\mathrm{Spin}}(X)$ is the $\mathfrak{spin}(d)$-valued 1-form

$$\omega^{\mathrm{Spin}} := \chi_*^{-1}\circ F^*\omega^{\mathrm{LC}}.$$

Proposition 18.9. *The spin connection of Definition 84 is indeed a connection on the* $\mathrm{Spin}(d)$*-principal bundle* $\mathrm{Fr}^{\mathrm{Spin}}(X)$*.*

Proof. The condition

$$(\chi_*^{-1}\circ F^*\omega^{\mathrm{LC}})(v^\sharp) = v, \qquad \forall v\in\mathfrak{spin}(d). \tag{18.20}$$

is left as an exercise.

As for the equivariance condition: for any $g\in\mathrm{Spin}(d)$,

$$\begin{aligned}
R_g^*(\chi_*^{-1}\circ F^*\omega^{\mathrm{LC}}) &= \chi_*^{-1}\circ(F\circ R_g)^*(\omega^{\mathrm{LC}}) \\
&= \chi_*^{-1}\circ(R_{\chi(g)}\circ F)^*(\omega^{\mathrm{LC}}) \qquad \text{(spin structure definition)} \\
&= \chi_*^{-1}\circ F^*(R_{\chi(g)}^*(\omega^{\mathrm{LC}})) \\
&= \chi_*^{-1}\circ F^*(\mathrm{Ad}_{\chi(g)^{-1}}(\omega^{\mathrm{LC}})) \quad (\omega^{\mathrm{LC}}\text{ is }\mathrm{SO}(d)\text{ equivariant}) \\
&= \mathrm{Ad}_{g^{-1}}\circ\chi_*^{-1}(F^*\omega^{\mathrm{LC}}). \qquad \text{(Prop. 15.5)}
\end{aligned}$$

□

[24] A less intrinsic definition: If X is a submanifold of Euclidean $\mathbb{R}^N$, then ∇^{LC} on TX is obtained by "orthogonally projecting" the trivial covariant derivative on $X\times\mathbb{R}^N$.

18.10 Spinor bundles and Clifford multiplication

Definition 85. Let X be a d-dimensional spin manifold. The *spinor bundle* $\mathcal{S} \to X$ is the associated Hermitian vector bundle

$$\mathcal{S} = \mathrm{Fr}^{\mathrm{Spin}}(X) \times_{\delta_n} S,$$

where $\delta_d : \mathrm{Spin}(d) \to S = \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}}$ is the spinor representation.

Recall that the spinor space S also admits an action of the Clifford algebra Cl_d.

Definition 86. *Clifford multiplication* is the map

$$\begin{aligned} \mathfrak{X}(X) \times \Gamma(\mathcal{S}) &\to \Gamma(\mathcal{S}) \\ (v, \psi) &\mapsto c(v)\psi \end{aligned}$$

defined, at each $x \in X$, by the formula

$$\big(c(v)\psi\big)(x) := [\tilde{\mathbf{e}}(x), c(\mathbf{v}(x)) \cdot \xi(x)], \tag{18.21}$$

where

- $\tilde{\mathbf{e}} : U \to \mathrm{Fr}^{\mathrm{Spin}}(X)$ is any local spin frame over $U \ni x$;
- $\xi : U \to S$ represents ψ with respect to the local spin frame $\tilde{\mathbf{e}}$;
- $\mathbf{v} : U \to \mathbb{R}^d$ represents v with respect to the local tangent frame $F \circ \tilde{\mathbf{e}}$;
- $c(\mathbf{v}(x))$ is the Clifford representation of $\mathbf{v}(x) \in \mathbb{R}^d$ on S.

The compatibility Proposition 18.8 is used for the following result.

Proposition 18.10. *Clifford multiplication is gauge-independent.*

Proof. We suppress the basepoint x for convenience. Suppose the spin frame $\tilde{\mathbf{e}}$ is modified by $g \in \mathrm{Spin}(d)$. Then

$$\psi = [\tilde{\mathbf{e}} \cdot g, \delta_d(g^{-1})(\xi)], \qquad v = [\mathbf{e} \cdot \chi(g), \chi(g^{-1})(\mathbf{v})].$$

The Clifford multiplication formula (18.21) becomes

$$\begin{aligned} c(v)\psi &= [\tilde{\mathbf{e}} \cdot g, c(\chi(g^{-1})(\mathbf{v}))\delta_d(g^{-1})(\xi)] \\ &= [\tilde{\mathbf{e}} \cdot g, \delta_d(g^{-1})c(\mathbf{v})\delta_d(g)\delta_d(g^{-1})(\xi)] && \text{(Prop. 18.8)} \\ &= [\tilde{\mathbf{e}}, c(\mathbf{v})\xi], && \text{(associated spinor bundle)} \end{aligned}$$

which is the same as when $\tilde{\mathbf{e}}$ was used.

□

Remark. If we have an (local) orthonormal coordinate tangent frame $\mathbf{e} = (\partial_1, \dots, \partial_d)$, then $ic(\partial_j)$ is multiplication by a Hermitian matrix squaring to 1. This matrix is called a *gamma matrix* in physics, and the compatibility (18.17) is the "transformation law" for gamma matrices.

18.11 (Twisted) Atiyah–Singer–Dirac operator

The spinor bundle $\mathcal{S}$ is a Hermitian vector bundle with structure group $\mathrm{Spin}(d)$. By Section 17.3, the spin connection ω^{Spin} induces a *spin covariant derivative* ∇^{Spin} on the spinor bundle $\mathcal{S}$. As the spin representation is unitary, ∇^{Spin} is metric compatible.

Definition 87. The *(Atiyah–Singer) Dirac operator* on a spin manifold X is the first-order differential operator on the spinor bundle $\mathcal{S}$ defined by

$$\begin{aligned} \not{D} = \not{D}^{(X)} : \Gamma(\mathcal{S}) &\to \Gamma(\mathcal{S}) \\ \psi &\mapsto \sum_{j=1}^{d} c(e_j) \nabla^{\mathrm{Spin}}_{e_j} \psi \end{aligned}$$

where $\{e_1, \dots, e_d\}$ is any local oriented orthonormal tangent frame[25].

Definition 87 appears to depend on local tangent frame choices, but it is actually gauge independent, due to Prop. 18.10. The global spin structure for X ensures that $\mathcal{S}$, and $\not{D}$ are globally well-defined.

Let $\mathcal{L} \to X$ be a Hermitian line bundle with unitary connection $\nabla^{\mathcal{L}}$. Then $\mathcal{S} \otimes \mathcal{L}$ is called a *twisted spinor bundle*, and it is equipped with the *tensor product covariant derivative*,

$$\nabla^{\mathrm{Spin},\mathcal{L}}(\psi \otimes \varphi) := (\nabla^{\mathrm{Spin}}\psi) \otimes \varphi + \psi \otimes (\nabla^{\mathcal{L}}\varphi), \qquad \psi \in \Gamma(\mathcal{S}),\ \varphi \in \Gamma(\mathcal{L}).$$

Clifford multiplication is defined on $\mathcal{S} \otimes \mathcal{L}$ by acting on the $\mathcal{S}$ factor.

Definition 88. The *twisted Dirac operator* on the twisted spinor bundle $\mathcal{S} \otimes \mathcal{L}$ is defined as

$$\not{D}_{\mathcal{L}} := \sum_{j=1}^{d} c(e_j) \nabla^{\mathrm{Spin},\mathcal{L}}_{e_j},$$

with $\{e_1, \dots, e_d\}$ any local oriented orthonormal frame (admitting a spin lift).

[25] Admitting a spin frame lift; work over a contractible $U \subset X$ to guarantee this.

Remark. For X the Euclidean/complex plane, the spin Dirac operator has the form (18.1), and squares to two copies of the "scalar" Laplacian. Recall the twisted version,

$$D_b = -2i \begin{pmatrix} 0 & \partial - \frac{b\bar{z}}{4} \\ \bar{\partial} + \frac{bz}{4} & 0 \end{pmatrix}$$

from Section 4.1.1. This squares to two copies of the Landau Hamiltonian, slightly shifted from one another. This is a clue that Dirac operator kernels ($\sim$indices) are of great importance in physics.

Another key property of the (twisted) Dirac operator is its formal self-adjointness, which is built into its geometric construction (Try proving this!). There are numerous mathematical and physical applications of (spin) Dirac operators, particularly their index theory, but these would require another course to explain.