3 Harmonic Oscillator and unbounded operators

3.1 Quantum particle on a line

Let us work on the Euclidean line as the spatial manifold $\mathcal{M}$. There are distinguished coordinate charts, called *Cartesian coordinates*, such that the distance between two points $p, p' \in \mathcal{M}$ is just given by $|x - x'|$, where x, x' are the corresponding Cartesian coordinates. This also determines a measure for the purposes of integrating functions on $\mathcal{M}$. (This is Lebesgue measure dx with respect to the Cartesian coordinate x.)

There is no room in 1D for orbital motion, and we do not have to worry about rotations and tangent spaces, or the concept of spinors. We shall also assume that there is no background magnetic field, so we do not have to worry about gauge fields, and we are allowed to pretend that quantum wave functions are elements of the Hilbert space $L^2(\mathcal{M})$ of square-integrable functions. There is only an electric potential function $V_t \in C^\infty(\mathcal{M}; \mathbb{R})$. We shall consider the case of time-independent potentials, and just write $V = V_t = V_0 \in C^\infty(\mathcal{M}, \mathbb{R})$.

So, in Cartesian coordinates $\mathcal{M} \cong \mathbb{R}$, the (time-independent) Schrödinger equation is

$$i\hbar \partial_t \psi_t = \left(-\frac{\hbar^2}{2m} \partial_x^2 + V \right) \psi_t \equiv H \psi_t,$$

for $\psi_t \in L^2(\mathcal{M}) \cong L^2(\mathbb{R})$. Instead of thinking of the equation as a PDE, we consider

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V$$

as a linear operator on the Hilbert space $L^2(\mathbb{R})$, and study the "linear ODE in Hilbert space",

$$i\hbar \frac{d}{dt} \psi_t = H \cdot \psi_t.$$

As mentioned, the solution is formally given by

$$\psi_t = e^{-iHt/\hbar} \psi_0,$$

where $\psi_0 \in L^2(\mathbb{R})$ is the initial wavefunction at $t = 0$. It seems that all we need to make sense of this solution is to "diagonalize" $H = SDS^{-1}$, and treat H as a "diagonal matrix D", then the above time-evolution would

be solved explicitly by eigenvector decomposition of ψ_0. As written, this prescription does not make mathematical sense, but it contains much of the correct physical ideas, informally called "matrix mechanics".

As a practical matter, diagonalization problems are generally intractable, but this is not what we want to focus on. There are more important conceptual issues with the naïve "diagonalization" idea:

- A general Hilbert space element $\psi \in L^2(\mathbb{R})$ does not have a classical derivative.
- The potential function V is typically an unbounded function.
- The derivative of a smooth $L^2(\mathbb{R})$ function may not remain square-integrable.
- Even after sorting out all the above domain issues, we need $e^{-iHt/\hbar}$ to be unitary operators (for the probability interpretation), "differentiable" with respect to t.

We will temporarily sweep all of these important issues under the carpet, and work formally only with the following nice subspace of wavefunctions, known as the *Schwartz space*,

$$\mathscr{S}(\mathbb{R}) = \left\{ \text{smooth } \psi : \mathbb{R} \to \mathbb{C} \, : \, \sup_{x \in \mathbb{R}} \left| x^\alpha \cdot \frac{d^\beta}{dx^\beta} \psi(x) \right| < \infty, \ \forall \, \alpha, \beta \in \mathbb{N} \right\}.$$

This is a dense subspace of $L^2(\mathbb{R})$, and is preserved under taking derivatives and multiplying by polynomial potential functions.

3.2 Quantum momentum and energy, formally

The *position operator* on the line, formally, the multiplication operator

$$X : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}), \qquad \psi \mapsto X\psi, \quad (X\psi)(x) = x \cdot \psi(x).$$

The *momentum operator* on the line is, formally, the differential operator

$$P : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}), \qquad \psi \mapsto \left(-i\hbar \frac{d}{dx} \right) \psi.$$

(The lazy way to treat $\hbar$ is to regard it as some number, or even set $\hbar = 1$. For real physical interpretation, the units are extremely important, and we will keep track of them.)

A key thing to remember is that both of these operators are *coordinate dependent*, and also depend on a metric (coordinate x really means "distance" x from some origin.). So even though one often says that they are position/momentum *observables*, we do not e.g., directly measure the "Cartesian coordinate" of a quantum particle. Rather, we might have set up some detectors[7] in a laboratory, and then with the help of rulers (giving a metric and Cartesian coordinates), label those detectors by convenient coordinates. Then we infer the "Cartesian coordinates of the quantum particle" based on which detector fires ("yes" or "no" outcome). Even more problematic is what the momentum operator corresponds to in a laboratory. You can look up the internet for interesting discussions on whether we can indeed "measure the momentum of quanta".

Anyway, the key feature of X, P is their failure to commute, and this failure will be characteristic of any sensible modification of the "canonical X, P" operators. Indeed, for the standard X, P given above, their *commutator* is especially simple,

$$[X, P] = i\hbar \cdot 1_{\mathscr{S}(\mathbb{R})}. \tag{3.1}$$

This equation is called the *canonical commutation relation*, and is sometimes interpreted as a kind of "uncertainty principle" for the position/momentum of a quantum particle — informally, we cannot simultaneously measure both to arbitrary precision.

Actually, Eq. (3.1) is *not* really canonical for several reasons (although there is a formulation, due to Weyl, which is a bit more canonical). One problem is the mismatch between the left and right sides. The right side can certainly be extended to all of $L^2(\mathbb{R})$, and seems like an "observable" which will give the definite outcome $i\hbar$ (since that is the unique eigenvalue). But the left side does not make sense for non-Schwartz functions. In practice, there is no such thing as an observable "$-i[X, P]$", and we certainly do not "measure $\hbar$" by performing a measurement corresponding to the mathematical operator $-i[X, P]$.

Notwithstanding the above practical issues with the operator P, it does deserve such a name because of its relation to the momentum concept in classical mechanics. Sketch:

[7] Detectors are macroscopic objects, like photomultipliers, cloud chambers, etc.

- The Hamiltonian operator H can be written as a "polynomial",

$$H \equiv H(X, P) = \frac{P^2}{2m} + V(X).$$

Note that despite X, P not commuting with each other, there is apparently no ordering issue with P and X, compositions like XP of PX do not appear. (But see Remark 1.) Nevertheless, H does not commute with $\frac{P^2}{2m}$, nor with $V(X)$. We will return to this observation shortly.

- Write

$$\psi = R \cdot e^{iS/\hbar},$$

where $S \equiv S(x)$ is a real-valued function. Rewriting out the Schrödinger equation with Hamiltonian $H(X, P)$ in terms of R, S, one finds that S satisfies a certain differential equation, which in the $\hbar \to 0$ limit, reduces to the so-called *Hamilton–Jacobi* equation[8] in classical mechanics for the classical Hamiltonian function $H(x, p) = \frac{p^2}{2m} + V(x)$.

- The variables x, p in the classical Hamiltonian $H(x, p)$ refer to the position coordinate and the momentum coordinate of a point in the classical phase space (the cotangent bundle of $\mathcal{M}$). The term $\frac{p^2}{2m}$ is the the classical *kinetic energy* function, while $V(x)$ is the classical *potential energy* function determining a (conservative) force field,

$$\mathbf{F}(x) = -\nabla V(x).$$

So $H(x, p)$ is the total energy of a classical point particle with mass m with momentum p and position x, when placed in the force field F.

- You may recall from classical physics courses that along any path which satisfies the classical equations of motion (e.g. Newton's law), the total energy is unchanged in time (when the force field comes from a potential function). This conservation law is why energy is a useful "intrinsic" quantity. The quantum mechanical "energy operator" is the Hamiltonian operator $H(X, P)$.

[8] There are several, more or less equivalent, formulations of classical mechanical equations of motions, including the most basic Newton's laws of motion, Lagrangian mechanics, Hamiltonian mechanics, Hamilton–Jacobi equations, etc. The Hamilton–Jacobi PDE formulation is probably the most difficult, but it gives a more direct relation to the Schrödinger PDE. You may consult the internet with these keywords to find the precise relation.

The formal assignment

$$\text{classical } (x,p) \;\longrightarrow\; (X,P)$$

is a useful rule-of-thumb, sometimes called Dirac's *quantization rule*. It allows us to convert a classical mechanical energy function into a quantum mechanical Hamiltonian operator.

3.3 1D Quantum harmonic oscillator

One of the most useful classical mechanical systems is the simple harmonic oscillator. As a concrete example, one considers a point particle of mass m, which is subject to a linear restoring force pointing towards a special equilibrium point,

$$F(x) = -kx = -\nabla(\tfrac{1}{2}kx^2).$$

The constant k is the *spring constant*, and the potential energy function is

$$V(x) = \frac{1}{2}kx^2.$$

Note that we are using the preferred Cartesian chart, where the special equilibrium point has coordinate 0.

It is convenient to write $k = m\omega^2$, where ω is the *angular frequency*. (Because one will find oscillating solutions to Newton's equation, with frequency ω.) So the classical total energy/Hamiltonian function is

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

The corresponding *quantum harmonic oscillator Hamiltonian* is the operator

$$H \equiv H(X,P) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 X^2.$$

The basic quantity with units of energy is $\hbar\omega$, so the numerical version is

$$\tilde{H} := \frac{1}{\hbar\omega}H = -\frac{\hbar}{2m\omega}\frac{d^2}{dx^2} + \frac{1}{2}\frac{m\omega}{\hbar}X^2.$$

Similarly, we rescale the coordinates[9] to $\tilde{x} = \sqrt{\frac{m\omega}{\hbar}}x$, and write

$$\tilde{X} = \sqrt{\frac{m\omega}{\hbar}}X = \text{mult. by } \tilde{x}, \qquad \tilde{P} = -i\frac{d}{d\tilde{x}} = -i\sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}. \tag{3.2}$$

[9]What we called x actually comes with a length scale (e.g. metre). The factor $\sqrt{\frac{m\omega}{\hbar}}$ is an inverse-length, so that $\tilde{x}$ is a pure number.

So

$$2\tilde{H} = \tilde{P}^2 + \tilde{X}^2, \tag{3.3}$$

and we seek the eigenvalues of this operator.

3.3.1 Supersymmetric solution

The eigenvalues of (3.3) can be found by the so-called *ladder operator*, or *supersymmetric* quantum mechanics method. The general idea is the following.

- The operator $\tilde{H}$ in (3.3) is second-order, so we could try to find a first-order "square root".
- Unfortunately, such a square root does not seem to exist.
- Actually, *two copies* of $\tilde{H}$, slightly shifted from each other, does admit a first-order square root. These two shifted copies of $\tilde{H}$ are called *supersymmetric partners*.

The supersymmetric partners for the harmonic oscillator example are found with the help of the canonical commutation relation,

$$-i[\tilde{X}, \tilde{P}] = 1.$$

Namely, we can write

$$\begin{aligned}
\begin{pmatrix} 2\tilde{H} - 1 & 0 \\ 0 & 2\tilde{H} + 1 \end{pmatrix} &= \begin{pmatrix} \tilde{P}^2 + \tilde{X}^2 + i[\tilde{X}, \tilde{P}] & 0 \\ 0 & \tilde{P}^2 + \tilde{X}^2 - i[\tilde{X}, \tilde{P}] \end{pmatrix} \\
&= \begin{pmatrix} (\tilde{X} - i\tilde{P})(\tilde{X} + i\tilde{P}) & 0 \\ 0 & (\tilde{X} + i\tilde{P})(\tilde{X} - i\tilde{P}) \end{pmatrix} \\
&=: \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} \\
&= \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}^2 =: D^2.
\end{aligned}$$

So $2\tilde{H} - 1$ and $2\tilde{H} + 1$ are partners, such that their direct sum admits a first-order square root $D = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}$. The operators

$$A^\dagger := \tilde{X} - i\tilde{P}, \qquad A := \tilde{X} + i\tilde{P},$$

are called the *creation/annihilation* operators, respectively.

All the manipulations above are valid on the Schwartz space. It is a simple matter to check that X and P are formally self-adjoint operators on the Schwartz space, meaning that

$$\langle A^\dagger \varphi | \psi \rangle_{L^2} = \langle \varphi | A\psi \rangle_{L^2}, \qquad \varphi, \psi \in \mathscr{S}(\mathbb{R}).$$

So the square root D is a formally self-adjoint operator. This is enough to guarantee that its eigenvalues (for Schwartz class eigenfunctions) are real, and that its eigenspaces for distinct eigenvalues are mutually orthogonal. The eigenvalues of D^2, thus of $\tilde{H}$ are actually controlled by D.

The key to finding the eigenvalues of $D^2 = A^\dagger A \oplus AA^\dagger$, is the following supersymmetry principle:

Exercise 3.1. Let $\mathcal{H}$ be a Hilbert space, and $A, A^\dagger : \mathrm{Dom} \to \mathrm{Dom}$ be two densely-defined operators which are formally adjoint to each other,

$$\langle A^\dagger \varphi | \psi \rangle = \langle \varphi | A\psi \rangle, \qquad \varphi, \psi \in \mathrm{Dom} \subseteq_{dense} \mathcal{H}.$$

Show that

- $A^\dagger A$ and $AA^\dagger$ are formally self-adjoint operators on Dom.
- The eigenvalues of $A^\dagger A$ and $AA^\dagger$ are non-negative.
- $A^\dagger A$ and $AA^\dagger$ have the same *non-zero* eigenvalues, with the same algebraic multiplicities.

The set of eigenvalues of an operator is its *point spectrum*, and is denoted $\sigma_{\mathrm{p}}(-)$. Exercise 3.3 says that

$$\sigma_{\mathrm{p}}(A^\dagger A) \setminus \{0\} = \sigma_{\mathrm{p}}(AA^\dagger) \setminus \{0\}. \tag{3.4}$$

But we also have $AA^\dagger = A^\dagger A + 2$, which has strictly positive eigenvalues. Substituting into (3.4), we get

$$\sigma_{\mathrm{p}}(A^\dagger A) \setminus \{0\} = \big(\sigma_{\mathrm{p}}(A^\dagger A) + 2\big) \setminus \{0\} = \sigma_{\mathrm{p}}(A^\dagger A) + 2.$$

Thus, $\sigma_{\mathrm{p}}(A^\dagger A)$ is a certain subset of $[0, \infty)$ satisfying

$$\sigma_{\mathrm{p}}(A^\dagger A) \setminus \{0\} = \sigma_{\mathrm{p}}(A^\dagger A) + 2.$$

There are only two possibilities:

$$\sigma_{\mathrm{p}}(A^\dagger A) = \begin{cases} \emptyset, \\ 2\mathbb{N}. \end{cases}$$

The second possibility occurs iff $A^\dagger A$ has a 0-eigenvalue iff A has non-trivial kernel. (Note that $AA^\dagger$, thus $A^\dagger$, does not have 0-eigenvalue, since its eigenvalues are strictly positive.) It is easy to find the 0-eigenfunctions ψ_0 of A:

$$0 = (A\psi)(\tilde{x}) = ((\tilde{X} + i\tilde{P})\psi_0)(\tilde{x}) = \Big(\tilde{x} + \frac{d}{d\tilde{x}}\Big)\psi_0(\tilde{x})$$
$$\Rightarrow \psi_0(\tilde{x}) \sim e^{-\tilde{x}^2/2}.$$

(The Gaussian ψ_0 is clearly in the Schwartz class.) So

$$\ker(A) = \mathrm{span}\{\psi_0\} \neq 0\{0\},$$

and therefore

$$\sigma_{\mathrm{p}}(\tilde{H}) = \frac{1}{2}\sigma_{\mathrm{p}}(A^\dagger A + 1) = \frac{1}{2}\big(2\mathbb{N} + 1\big) = \mathbb{N} + \frac{1}{2}.$$

Returning to physics, and recalling the energy rescaling $H = \hbar\omega\tilde{H}$, we have found that

$$\sigma_{\mathrm{p}}(H) = \big(\mathbb{N} + \frac{1}{2}\big)\hbar\omega.$$

In other words, a quantum harmonic oscillator has a *discrete* set of energy eigenvalues which are evenly spaced by an amount $\hbar\omega = h(\omega/2\pi) = h\nu$. Therefore, we would expect that it absorbs/emits electromagnetic radiation of some sharp frequency $\nu = \omega/2\pi$.

Remark. What kind of actual physical system is reasonably modelled by a simple harmonic oscillator H? Perhaps a classical "ball attached to spring"? Such a "classical" system has ν on the scale of Hertz ($\frac{1}{\mathrm{second}}$), and the associated photon would have extremely small energy $h\nu \sim 10^{-34}$ Joules. This energy resolution is way beyond any available measuring device.

Instead, one typically uses quantum harmonic oscillator Hamiltonians to model vibrations of microscopic molecules, e.g., a diatomic one. Classically, the positively-charged nuclei repel each other, but they also attract

the negatively-charged electrons of the other. Overall, there is some equilibrium configuration (for the centre-of-mass) which minimizes the complicated potential energy function. One approximates this (unknown) potential function by some quadratic one (Taylor expansion), $\frac{1}{2}kx^2 = \frac{1}{2}m\omega^2$. The kind of "spring constant k", or angular frequency ω, in this context is surprisingly large, so that the energy jumps in the quantum mechanical analysis are quite large. These would correspond to photons with specific energy, or electromagnetic radiation with certain frequencies. These frequencies are detected by infrared spectroscopy.

Remark. The quantum state ψ_0 with definite energy $E = \hbar\omega$ is a Gaussian with support on the whole line. In particular, there is a non-zero probability of detecting it at $[x-\epsilon, x+\epsilon]$, no matter how large x is. In contrast, a classical path solving the classical equations of motion has a conserved energy, say E. This path cannot reach the point x, if the potential energy $V(x)$ there exceeds E, otherwise the kinetic energy would be negative.

So one learns that a quantum state with energy E is actually able to escape out of its classical potential well. The issue, as already pointed out previously, is that the Hamiltonian operator does not commute with either the position operator, or the momentum operator alone. Likewise, it is senseless to say that an energy-E quantum state "really" has a certain position, or momentum, or potential energy, or kinetic energy. Those are really concepts that only make sense for a classical path.

Remark 1. Another mysterious feature is that the lowest energy eigenfunction seems to have strictly positive energy $\frac{1}{2}\hbar\omega$. Whereas classically, the lowest energy state is the constant path at the bottom of the potential well, and this solution has zero total energy. A quantum state with this kind of classical feature of definite position and definite (zero) momentum does not exist.

The "zero-point energy" $\frac{1}{2}\hbar\omega$, however, is mostly a mathematical artifact. Energy values, even classically, lies in an $\mathbb{R}$-torsor, because the potential energy function is indeterminate up to an overall constant (it is an indefinite integral). In general, if ψ_t solves the Schrödinger equation for Hamiltonian H, then $e^{-iE_0t/\hbar}\psi_t$ solves the Schrödinger equation for Hamiltonian $H + E_0$. The extra phase $e^{-iE_0t/\hbar}$ applies uniformly to all wavefunctions (at each given time t), and transition amplitudes between wavefunctions are unaffected by such a modification. The multiplication by $e^{-iE_0t/\hbar} \in \mathrm{U}(1)$ is an example of a global (x-independent) gauge transformation, to be contrasted with the more interesting concept of local U(1)-gauge transformations.

Indeed, once we allow for complex-valued functions, we can factorize the classical harmonic oscillator Hamiltonian function as

$$2\tilde{H}'(\tilde{x},\tilde{p}) = \tilde{p}^2 + \tilde{x}^2 = (\tilde{x} - i\tilde{p})(\tilde{x} + i\tilde{p}).$$

Then the corresponding quantum Hamiltonian should be

$$2\tilde{H}' = (\tilde{X} - i\tilde{P})(\tilde{X} + i\tilde{P}) = A^\dagger A = \tilde{X}^2 + \tilde{P}^2 - 1 = 2\tilde{H} - 1.$$

This alternative Hamiltonian $\tilde{H}'$ has lowest eigenvalue being 0.

Should we have used H or H'? Either works, and this exhibits an inherent ambiguity in the formal quantization procedure of turning the commuting coordinate functions (x, p) into the *noncommuting* operators X, P.

The Gaussian eigenfunction ψ_0 has various names in physics: *vacuum state*, *harmonic oscillator ground state*, *supersymmetry-breaking ground state*, *zero mode*, etc. Actually, the supersymmetry method determines not only the point spectrum, but also all the eigenspaces, as the following exercise reveals.

Exercise 3.2. For the $A, A^\dagger$ in the harmonic oscillator example, show that

- $\ker(A^\dagger) = \{0\}$.
- For $n \geq 1$, $\psi_n \in \mathscr{S}(\mathbb{R})$ is an eigenvector of $AA^\dagger$ with eigenvalue n iff $A^\dagger\psi_n \in \mathscr{S}(\mathbb{R})$ is an eigenvector of $A^\dagger A$ with eigenvalue n.
- For $n \geq 0$, the $(n+\frac{1}{2})$-eigenspace of H_{SHO} is spanned by $(A^\dagger)^n\psi_0$, where ψ_0 is the Gaussian $\psi_0(\tilde{x}) = e^{-\tilde{x}^2/2}$.

So far, we have calculated very algebraically. The main missing ingredient is whether the eigenspaces of the harmonic oscillator Hamiltonian are enough to span the whole Hilbert space $L^2(\mathbb{R})$, at least approximately. If not, there are actually some other quantum states that are unaccounted for, and we cannot make conclusions such as $\hbar\omega$ being the only available energy transition step. Mathematically, this is a question about the *completeness* of the eigenbasis for formally self-adjoint operators, and a rigorous answer requires considerable tools from functional analysis.

For some context, there is a very important Hamiltonian, with potential function $V \equiv V(r) \sim \frac{1}{r}$, where r is a radial coordinate. This is the *Coulomb*

potential, and is relevant for describing, e.g., the electron of a hydrogen atom. It turns out that the eigenspaces of the Coulomb Hamiltonian can also be explicitly found, but the eigenspaces are *not* sufficient to span the full Hilbert space of wavefunctions! This has important physical consequences: the missing wavefunctions describe "scattering states" of the electron, when it has acquired enough energy to escape from the nucleus.

3.4 Index

The "square root" D is actually an example of a *Dirac operator*, and such operators have very structured spectrum, related to the geometry of the space on which it is defined. This is analogous to how the eigenvalues of the classical Laplacian relates to the underlying Riemannian manifold. So Dirac operator spectrum has something to do with $\sqrt{\text{Riemannian}}$ geometry, also known as spin geometry. These concepts will be covered later in the course.

For now, here is an exercise revealing some general properties of eigenvalues of such operators.

Exercise 3.3. Let $A, A^\dagger : \text{Dom} \to \text{Dom}$ be two densely-defined operators which are formally adjoint to each other.

1. Show that $D := \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}$ is symmetric on $\text{Dom} \oplus \text{Dom}$.
2. Show that
$$\ker(D^2) = \ker(D) = \ker(A) \oplus \ker(A^\dagger).$$
3. If $\binom{\psi_1}{\psi_2}$ is an eigenvector of D with non-zero eigenvalue λ, show that $\psi_1 \neq 0 \neq \psi_2$.
4. Show that the point spectrum $\sigma_{\text{p}}(D)$ is symmetric about 0, i.e.,
$$\sigma_{\text{p}}(D) = -\sigma_{\text{p}}(D).$$

In the case of the harmonic oscillator, Exercise 3.3 shows that the *nonzero* eigenvalues of D occur in $\pm\lambda$ pairs. Furthermore, the eigenfunctions $\binom{\psi_1}{\psi_2}$ are nonzero in both components. These $\pm\lambda$ eigenfunctions span the $\lambda^2 > 0$ eigenspaces of $A^\dagger A \oplus AA^\dagger$. However, the kernel

$$\ker(D) = \ker(A) \oplus \ker(A^\dagger) = \ker(A) = \ker(A^\dagger A)$$

is unpaired, and occurs only in the first component of $\mathcal{S}(\mathbb{R}) \oplus \mathcal{S}(\mathbb{R})$.

In general, the *supersymmetric index* of $D = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}$ is defined as the integer

$$\mathrm{Index}(D) := \dim \ker(A) - \dim \ker(A^\dagger). \tag{3.5}$$

Mathematically, this is known as the *Fredholm index* of A. For the harmonic oscillator, we had just computed $\mathrm{Index}(D) = 1$.

The reason why the index concept is of great importance in modern physics, is its *stability*, and *homotopy invariance*. The idea is that if the supersymmetry structure is preserved, then we can only create/annihilate nonzero eigenvalues in $\pm$ pairs, by removing/adding a null space of A and a null space of $A^\dagger$ simultaneously. So the supersymmetric index is invariant against such perturbations. In particular, the supersymmetric ground state ψ_0 does not easily disappear when we perturb a supersymmetric model.