4 Recap of basic functional analysis

4.1 Some reasons to control infinities

4.1.1 Preview of quantum Hall effect

Consider the Euclidean plane manifold, with Cartesian coordinates x, y. The Laplacian on functions $\mathbb{R}^2 \to \mathbb{C}$ is usually $\Delta = \partial_x^2 + \partial_y^2$. Its eigenfunctions are the plane waves,

$$(x, y) \mapsto e^{i(p_x x + p_y y)}, \qquad (p_x, p_y) \in \mathbb{R}^2,$$

with eigenvalues $-(p_x^2 + p_y^2) \leq 0$. These are not square-integrable.

Now consider a magnetic field of uniform strength $b > 0$ applied perpendicularly to the plane. (Here, we really mean $b\times$ some unit of magnetic field strength, say the SI unit Tesla.) The "free" Laplacian for an electron of charge $-e$ is modified to

$$\Delta_b = \big(\partial_x + \frac{ieby}{2\hbar}\big)^2 + \big(\partial_y - \frac{iebx}{2\hbar}\big)^2.$$

We will see that the reason for modifying the partial derivative operators is geometric and physical. For now, take it as a model proposal that the Hamiltonian for a "free" electron moving in a plane and subject to the above-mentioned magnetic field, is

$$H_b = -\frac{\hbar^2}{2m}\Delta_b.$$

We will see that the eigenvalues of H_b are

$$\sigma_{\mathrm{p}}(H_b) = (\mathbb{N} + \tfrac{1}{2})\hbar\frac{eb}{m}, \tag{4.1}$$

which is that of a 1D harmonic oscillator.

For simplicity, we shall set $e, m, \hbar$ to 1, and use complex coordinates $z = x + iy$. The holomorphic and antiholomorphic derivatives are

$$\partial = \frac{1}{2}(\partial_x - i\partial_y), \qquad \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y).$$

Then

$$2H_b = -\big(\partial_x + \frac{iby}{2}\big)^2 + \big(\partial_y - \frac{ibx}{2}\big)^2.$$

The trick to finding the eigenvalues is to consider the following operator,

$$D_b = -2i \begin{pmatrix} 0 & \partial - \frac{b\bar{z}}{4} \\ \bar{\partial} + \frac{bz}{4} & 0 \end{pmatrix}.$$

As an exercise, check that

$$D_b^2 = \begin{pmatrix} 2H_b - b & 0 \\ 0 & 2H_b + b \end{pmatrix}$$

As in the harmonic oscillator case, the supersymmetric argument reveals that

$$\sigma_{\mathrm{p}}(2H_b - b) \setminus \{0\} = \sigma_{\mathrm{p}}(2H_b + b) = \sigma_{\mathrm{p}}(2H_b - b) + 2b,$$

and therefore,

$$\sigma_{\mathrm{p}}(2H_b - b) = 2b\mathbb{N} \;\Leftrightarrow\; \sigma_{\mathrm{p}}(H_b) = (\mathbb{N} + \frac{1}{2})b.$$

Putting physical units back in, we get (4.1) as claimed. These eigenvalues are called *Landau levels*.

We learn that a uniform perpendicular magnetic field provides another mechanism that quantizes the possible energy eigenvalues (in this simple model) to a discrete set. The physical manifestation, however, is rather different from the harmonic oscillator case, where one measured the associated frequency corresponding to the energy jumps. Instead, one finds that a certain *macroscopic* property — the electrical Hall conductance[10] — is *exactly* quantized to integer multiples of $\frac{e^2}{h}$. Furthermore, this property is unaffected by a large variety of disturbances and imperfections in the 2D sample. This astonishing phenomenon is called the *Quantum Hall Effect*. It was discovered by Klaus von Klitzing in 1980, and he was awarded the Nobel prize in 1985.

One of the key differences with the harmonic oscillator case is the nature of the eigenspaces. The eigenspace of H_b for the Lowest Landau Level $\frac{b}{2}$ is precisely the kernel of D_b,

$$\begin{aligned} \ker(D_b) &= \ker(\bar{\partial} + \frac{bz}{4}) \oplus \underbrace{\ker(\partial - \frac{b\bar{z}}{4})}_{=0} \\ &= \overline{\mathrm{span}}\big\{ z^m e^{-m|z|^2/4} \,:\, m \in \mathbb{N} \big\}. \end{aligned} \tag{4.2}$$

[10] Actually one measures the resistivity, which is roughly speaking, the reciprocal of the conductance.

These basis eigenfunctions are all square-integrable. In other words, the Lowest Landau Level eigenspace is *infinitely degenerate*. Mathematically, this kind of spectrum is not really "discrete", but "essential". Essential spectrum has a certain "rigidity" to it, which makes it harder to perturb away than discrete spectrum like finite-multiplicity eigenvalues.

In *solid-state* (or *condensed matter*) physics, one typically studies some macroscopic collection of quantum mechanical particles, not just atoms or molecules. The simplest situation is when the electrons are assumed not to interact with each other. As a general rule (Pauli exclusion principle), they will then "occupy" the energy eigenstates of the Hamiltonian operator one-by-one, in ascending order. By adjusting b, it is possible to arrange for "approximately" one Landau level to be "occupied" with electrons[11]. The collective "state" is described by the projection operator P onto the occupied part of the Hilbert space.

So consider P to be the orthogonal projection operator onto the subspace (4.2). (This describes the state where the Lowest Landau Level is filled with electrons.) The Hall conductance may be argued from physics considerations to be given by the formula

$$2\pi i \cdot \mathrm{Tr}[PXP, PYP] \qquad \Big(\times \frac{e^2}{\hbar} \Big).$$

The symbol $\mathrm{Tr}(-)$ is the *trace*, which roughly means to sum over all eigenvalues. The symbols X, Y refer to certain "regularized" x and y coordinate operators. You might be surprised to find out that the above trace-of-commutator is possibly nonzero. Such a thing can never happen on a finite-dimensional vector space. (Why?)

The above trace formula makes rigorous mathematical sense for a large class of P similar to the Landau level eigenprojection, but quite a bit of functional analysis is needed, since infinite sums are involved. Remarkably, it is possible to *prove* that this trace must be an integer! (In fact, it is equal to 1 for the Landau level case.) All proofs of such a result are highly non-trivial, requiring some *index theory*.

In other words, physicists' arguments and experiments suggest a very surprising expectation that a certain a priori $\mathbb{C}$-valued trace must actually be

[11]Here, there is a finite density of electrons, but over an infinite plane, there are "infinitely many" electrons. The appearance of infinity in both cases is in the mathematical modelling, of actual large but finite volume samples.

integer-quantized. This is very similar to the famous Gauss–Bonnet theorem, stating that the integral of the scalar curvature of a closed surface equals 2π times of its topological Euler characteristic. Both these results are examples of *index theorems*.

4.1.2 Discontinuous observables

We had emphasized that the Schrödinger equation is, abstractly speaking, a linear ODE in Hilbert space. That is, quantum mechanical Hamiltonian operators are certain linear operators H meant to "generate" the time evolution by exponentiation, $U_t = e^{-iHt/\hbar}$.

The basic (useful!) examples of the free Hamiltonian and the harmonic oscillator already involves differential operators, and we mentioned that they may not map Hilbert space elements to Hilbert space elements, not even for general smooth elements in L^2. The basic difficulty is the unboundedness, or discontinuity, of such operators.

For example, consider the Hilbert space $\mathcal{H} = L^2([0, 2\pi])$. The functions

$$e_n : x \mapsto \frac{1}{\sqrt{2\pi}} e^{inx}, n \in \mathbb{Z}$$

are orthonormal. We can apply the momentum operator to each e_n,

$$-i\frac{d}{dx} : e_n \mapsto n e_n,$$

and extend this to *finite* linear combinations without issue.

But suppose we have an infinite sum, such as

$$\sum_{n\in\mathbb{N}} \frac{1}{n} e_n,$$

which has square-summable coefficients, thus $\psi \in \mathcal{H}$. If we tried to commute $\sum_{n\in\mathbb{N}}$ past $-i\frac{d}{dx}$, we would get

$$-i\frac{d}{dx} \sum_{n\in\mathbb{N}} \frac{1}{n} e_n \stackrel{?}{=} \sum_{n\in\mathbb{N}} -i\frac{d}{dx}\left(\frac{1}{n} e_n\right) \stackrel{?}{=} \sum_{n\in\mathbb{N}} e_n, \tag{4.3}$$

but the right side does not converge in $\mathcal{H}$. The problem is that there is no uniform bound on how $-i\frac{d}{dx}$ scales the Hilbert space norm. This prevents us

from naïvely extending the definition of $-i\frac{d}{dx}$ to all of $\mathcal{H}$ by "continuity". Of course, we could extend it *dis*continuously to all of $\mathcal{H}$, but this is probably a useless and non-canonical proposal.

Arguably, the unboundedness is a mathematical artifact. So a reasonable idea would be to fix the problem by taking the "bounded transform" of $-i\frac{d}{dx}$. This should be the operator given by the formula

$$\left(-i\frac{d}{dx}\right)_{\text{bounded}} : e_n \mapsto \frac{n}{\sqrt{1+n^2}} e_n. \tag{4.4}$$

This operator decreases the norm of all the basis elements e_n, so we may extend it operator to arbitrary elements $\psi \in \mathcal{H}$ by linearity and continuity.

However, this formal procedure actually assumes that we know the eigenfunctions of $-i\frac{d}{dx}$ (which are e_n). For some other discontinuous linear operator T, how would we even achieve this rescaling? This discussion indicate that formal symbolic manipulations like (4.4) cannot be separated from the problem of diagonalization (spectral theory).

4.1.3 Failure of self-adjointness

On the Euclidean line, the momentum operator $P = -i\frac{d}{dx}$ is easily checked to be formally self-adjoint on the subspace $C_c^\infty(-\infty,\infty) \subseteq L^2(-\infty,\infty)$ of compactly supported smooth functions. Just integrate-by-parts: for $\varphi, \psi \in C_c^\infty(-\infty,\infty)$, supported within some compact subinterval of (a,b),

$$\begin{aligned}
\left\langle -i\frac{d}{dx}\varphi \middle| \psi \right\rangle_{L^2(-\infty,\infty)} &= \int_a^b i\overline{\varphi'(x)}\psi(x)\,dx \\
&= \overline{\varphi}\cdot\psi\Big|_a^b - \int_a^b \overline{\varphi(x)} i\psi'(x)\,dx \\
&= \left\langle \varphi \middle| -i\frac{d}{dx}\psi \right\rangle_{L^2(-\infty,\infty)}
\end{aligned}$$

There just remains the technical problem of making it self-adjoint in a precise way.

Supposing that this technicality is resolved, and that consequently, the 1-parameter group

$$t \mapsto e^{itP} = e^{it(-id/dx)} = e^{td/dx}$$

makes sense. Let $f : \mathbb{R} \to \mathbb{C}$ be a smooth function. Applying the "exponential" series formally to f, we get

$$\begin{aligned}(e^{td/dx}f)(x) &= f(x) + tf^{(1)}(x) + \frac{t^2}{2}f^{(2)}(x) + \cdots \\ &= f(x+t).\end{aligned} \tag{4.5}$$

So it seems that $e^{td/dx}$ is the left translation-by-t operator, for each $t \in \mathbb{R}$. Of course, this kind of formal series argument does not make sense for non-smooth f (there are plenty of such functions in L^2), although the translation-by-t operators do make sense on L^2, and are furthermore unitary operators. It appears that $t \mapsto e^{itP}$ is a "differentiable" map with values in unitary operators, such that its "derivative" is the self-adjoint operator P. We can even consider the "Schrödinger" equation

$$i\frac{d}{dt}\psi_t = -P\psi_t$$

to be "solved" by applying the "time-evolution group" e^{itP} to any initial wavefunciton $\psi_0 \in L^2(-\infty, \infty)$. All these expectations turn out to be true, so one is "justified" in arguing formally.

Next, consider the "momentum operator" $P - i\frac{d}{dx}$ on the Euclidean half-line, $\mathbb{R}_+ = (0, \infty)$. The same integration-by-parts argument shows that it is formally self-adjoint on the domain

$$C_c^\infty(0, \infty) \subseteq L^2(0, \infty).$$

Suppose that the same exponential series argument makes sense, then we would expect that P exponentiates to the 1-parameter group $t \mapsto e^{itP}$ of unitary operators on $L^2(0, \infty)$. The calculation (4.5) would again suggest that e^{itP} is left translation-by-t. For positive t, this translation is not a unitary operator! Any wavefunction will eventually "hit the wall" and we will "lose probability".

We could try to fix this problem by observing that

$$\Big\langle -i\frac{d}{dx}\varphi \Big| \psi \Big\rangle_{L^2(0,\infty)} = \Big\langle \varphi \Big| -i\frac{d}{dx}\psi \Big\rangle_{L^2(0,\infty)} \tag{4.6}$$

holds for all $\psi \in C_c^\infty(0, \infty)$ and all $\varphi \in C_c^\infty[0, \infty)$. That is, we enlarge the domain by allowing φ to be non-vanishing at the boundary $x = 0$. If this

idea is entertained, we might as well consider the slightly bigger domain of smooth L^2 functions with L^2-derivative. Unfortunately, this enlarged domain will be too large to maintain formal self-adjointness (If $\psi(0) \neq 0$ as well, we will violate (4.6)). Fine, but what is so important about (formal) self-adjointness? Well, the enlarged domain includes the eigenfunction $\psi(x) = e^{-x}$ with *imaginary* eigenvalues!

It seems that there is some fundamental problem with making the "momentum operator" on $(0, \infty)$ self-adjoint, related to the imbalance between "left-moving" and "right-moving". We will see that the "momentum operator" indeed suffers from such a "chiral anomaly". The precise statement is that P on $(0, \infty)$ does not admit any self-adjoint realization, whatsoever. Physically speaking, this problem should prompt you to reconsider whether momentum really makes sense generally as an observable in quantum mechanics.

Remark. Classically speaking, although the manifolds $(-\infty, \infty)$ and $(0, \infty)$ are diffeomorphic, they are very different as Riemannian manifolds. On $(-\infty, \infty)$, there are geodesics (straight paths with constant velocity) which extend to all time. But on $(0, \infty)$, the geodesics are incomplete.

4.1.4 Some functional analytic themes to be covered

The main aims of the functional analytic part of the course are thus:

- Get a good foundation of basics of Hilbert spaces.
- Some Banach spaces for the concept of spectrum.
- Understand how unbounded operators are defined and manipulated, and why.
- Appreciate the depth of the concept of self-adjointness, and the spectral theorem.
- See how the theory of self-adjoint extensions apply to some simple physical situations.
- Study the concept of traces and determinants of operators on Hilbert spaces.

- Combine the above with differential geometry to incorporate the inherent U(1)-gauge theory, coupling quantum charged particles to (classical) electromagnetic field.

4.2 Basic notions

4.2.1 Inner products, norms, metrics

Let V be a complex inner product space. Our convention is that the inner product $\langle \cdot | \cdot \rangle$ is complex linear in the second argument and antilinear in the first argument. The associated *norm* is

$$||v|| = \sqrt{\langle v|v\rangle}, \qquad v \in V,$$

and the metric is

$$d(v, v') = ||v' - v||, \qquad v, v' \in V.$$

As with any metric space, the balls $B_r(v)$ of points with distance $< r$ from $v \in V$ can be defined, and these balls generate the open sets of a topology on V.

If v, v' are orthogonal, i.e., $\langle v|v'\rangle = 0$, then the *Pythagoraean identity* holds,

$$||v + v'||^2 = ||v||^2 + ||v'||^2.$$

The norm associated to the inner product satisfies the *parallelogram identity*,

$$||v + v'||^2 + ||v - v'||^2 = 2\left(||v||^2 + ||v'||^2\right).$$

An inner product can be recovered from its associated norm via the *polarization* identity (Exercise),

$$\langle v|v'\rangle = \frac{1}{4}\left(||v + v'||^2 - ||v - v'||^2\right) + \frac{i}{4}\left(||v - iv'||^2 - ||v + iv'||^2\right). \quad (4.7)$$

4.2.2 Hilbert and Banach spaces

Definition 3. A *Banach space* V (respectively, *Hilbert space* $\mathcal{H}$) is a normed vector space (respectively, inner product space) over $\mathbb{C}$, for which the metric is complete. (Cauchy sequences are convergent.)

4.2.3 ℓ^p spaces

Let c_{00} be the vector space of sequences $x : \mathbb{N} \to \mathbb{C}$ with finitely many non-zero entries. For each $1 \leq p < \infty$, we may define the ℓ^p norm

$$||x||_p = \left(\sum_{i \in \mathbb{N}} |x_i|^p \right)^{1/p}. \tag{4.8}$$

For $p = \infty$, the ℓ^∞ norm is

$$||x||_\infty = \sup_{i \in \mathbb{N}} |x_i|.$$

For $p = 2$, an inner product can be defined by

$$\langle x | x' \rangle = \sum_{i \in \mathbb{N}} \overline{x_i} x'_i, \tag{4.9}$$

and this induces the ℓ^2 norm.

Example 4.1. The sequence space c_{00} equipped with the ℓ^p norms are not Banach spaces (Exercise).

Example 4.2. For $1 \leq p < \infty$, the space ℓ^p is defined as the subset of sequences $x : \mathbb{N} \to \mathbb{C}$ such that

$$\sum_{i \in \mathbb{N}} |x_i|^p < \infty.$$

Then Eq. (4.8) defines a norm on ℓ^p, making it a *Banach space* (Exercise). Similarly, ℓ^2 is a Hilbert space, under the inner product (4.9).

For sequences indexed by $\mathbb{Z}$ instead of $\mathbb{N}$, we write $\ell^p(\mathbb{Z})$.

4.2.4 L^p spaces

The theory of measure spaces and Lebesgue integration are dealt with in detail in standard analysis textbooks. We give a summary here.

Recall that the general definition of a σ-algebra on a set X is collection σ of subsets of X such that

- $X \in \Sigma$,
- $A \in \Sigma$ iff $A^c = X \setminus A \in \Sigma$,

- If $A_1, A_2, \ldots \in \Sigma$, then $A = \cup_{i\in\mathbb{N}} A_i \in \Sigma$.

A σ-algebra determines which subsets of X are *measurable*. A function $f : X \to Y$ between spaces with σ-algebras is said to be *measurable*, if the preimages of measurable subsets are measurable.

Let X be equipped with a σ-algebra Σ. A *measure* on (X, Σ) is an assignment

$$\mu : \Sigma \to [0, \infty]$$

such that

- $\mu(\emptyset) = 0$,
- $\mu\big(\bigsqcup_{i\in\mathbb{N}} A_i\big) = \sum_{i\in\mathbb{N}} \mu(A_i)$, for disjoint $A_i \in \Sigma$.

The triple (X, Σ, μ) is called a *measure space*.

It is often convenient to admit all subsets of μ-measure-0 sets as measurable sets. Write $\tilde{\Sigma}$ for the σ-algebra generated by Σ and these extra sets. Elements of $\tilde{\Sigma}$ have the form $A \cup N$ where $A \in \Sigma$ and N is one of the extra sets. Then μ is *completed* to a measure $\tilde{\mu}$ by taking $\tilde{\mu}(A \cup N) = \mu(A)$.

Definition 4. If X is a topological space, then the *Borel σ-algebra* is the smallest σ-algebra containing all the open sets of X. A *Borel measure* is a measure on a Borel σ-algebra. If X is a metric space, *metric measure* is a Borel measure for the topology induced by the metric.

Example 4.3. Consider $\mathbb{R}$ with the standard metric, topology, and Borel σ-algebra. We may assign $\mu(a, b] = b - a$ for any $a, b \in \mathbb{R}$, and extend this to a measure on the Borel σ-algebra. The completion is the *Lebesgue* measure on $\mathbb{R}$. On $\mathbb{R}^d$, the Lebesgue measure is the completion of the n-fold product of the Lebesgue measure on $\mathbb{R}$.

Example 4.4. If X is a (path-connected) Riemannian manifold, we can regard it as a metric space, and there is a canonical way to get a metric measure. This will be explained in lectures on Riemannian volume forms/densities.

The *Lebesgue integral* is a generalization of the Riemann integral. Furthermore, it can deal with functions on general measure spaces. We refer to standard textbooks, e.g. Rudin's Real and Complex Analysis, Folland's Real Analysis, for full details of how $\int_X f \, d\mu$ is defined. Here is a brief summary.

A measurable function s is *simple* if its range is just a finite set of points $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. If we set $A_i := \{t \in X \,:\, s(t) = \alpha_i\}$ and write χ_{A_i} for the characteristic function of A_i, then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}.$$

When s is $\mathbb{R}_{\geq 0}$-valued, we define

$$\int_X s \, d\mu := \sum_{i=1}^{n} \alpha_i \mu(A_i).$$

For a general $[0, \infty]$-valued measurable function f, we define

$$\int_X f \, d\mu := \sup_{0 \leq \text{simple } s \leq f} \int_X s \, d\mu.$$

A complex-valued measurable function f is *integrable* with respect to μ if $\int_X |f| \, d\mu < \infty$. In that case, the integral $\int_X f \, d\mu \in \mathbb{C}$ is obtained by integrating separately the positive/negative parts of the real/imaginary parts of f, then assembling the four non-negative results back into a single complex number.

An important point is that many properties of functions are stated as *μ-almost everywhere* (μ-a.e.) properties. For example, measurable functions f, g are equal μ-a.e. if the subset on which they differ has μ-measure zero.

Definition 5. On a space X with measure μ, and for $1 \leq p < \infty$, the space $L^p(X) = L^p(X, \mu)$ is defined to be the μ-a.e. equivalence classes of measurable functions $f : X \to \mathbb{C}$, such that

$$||f||_p = \left(\int_X |f|^p \, d\mu \right)^{1/p} < \infty.$$

For $p = 2$, we define the inner product,

$$\langle f | g \rangle = \int_X \overline{f} g \, d\mu,$$

which induces the norm $|| \cdot ||_2$. The space $L^\infty(X)$ is the set of μ-a.e. equivalence classes of μ-a.e. bounded functions $f : X \to \mathbb{C}$, and we define

$$||f||_\infty = \operatorname{ess\,sup}|f| = \inf\{\nu \in \mathbb{R}_{\geq 0} \,:\, |f(x)| \leq \nu \text{ holds } \mu\text{-a.e.}\}.$$

Using the basic monotone/dominated convergence theorems for the Lebesgue integral, one shows that the $L^p(X)$ are Banach spaces, and that $L^2(X)$ is a Hilbert space. If we used the Riemann integral (for $X = \mathbb{R}^d$), we would not have enough p-integrable functions to achieve completeness.

Remark. Although the process of completion seems like a mere mathematical construct, its real purpose is to control approximations, so that we may understand whether a certain mathematical model is reliable for describing a class of physical systems of interest.