# 5 Structure of (operators on) Hilbert space

## 5.1 Orthogonal projection and complement

If $V$ is an inner product space, and $W \subset V$ is a linear subspace, the *orthogonal complement* of $W$ is

$$W^\perp := \{v \in V \,:\, \langle v|w\rangle = 0 \;\; \forall w \in W\}.$$

Clearly $W^\perp$ is a subspace, and $W \cap W^\perp = \{0\}$.

*Exercise* 5.1. For a subspace $W \subset V$, check the following properties of the orthogonal complement (Exercise):

- $W \subset W^{\perp\perp}$.
- If $W \subset W'$, then $W'^\perp \subset W^\perp$.
- $W^\perp = W^{\perp\perp\perp}$.

Suppose $V$ is finite-dimensional, and $W$ is a subspace. Then, for each $v \in V$, there is a unique $w_v \in W$ which is nearest to $v$. Furthermore, the vector $v - w_v$ lies in $W^\perp$. Thus we have an *orthogonal* direct sum,

$$V = W \oplus W^\perp, \qquad v = w_v + \big(v - w_v\big). \tag{5.1}$$

However, if $V$ is infinite-dimensional, the "point in $W$ nearest to $v$" may not exist. This issue is fixed when $V$ is a Hilbert space $\mathcal{H}$ and $W$ is a closed subspace.

**Lemma 5.1.** *Let $W \subset \mathcal{H}$ be a closed subspace of a Hilbert space, and $v \in \mathcal{H}$. There exists a unique $w_v \in W$ such that*

$$||v - w_v|| = \mathrm{dist}(v, W) \equiv \inf_{w \in W} ||v - w||.$$

*Furthermore, $w_v$ is the unique vector in $W$ such that $v - w_v \in W^\perp$.*

*Proof.* Write $\delta = \inf_{w\in W} ||v - w||$. Let $\{w_n\}$ be a sequence in $W$ with

$||v - w_n|| \to \delta$. By the parallelogram identity,

$$\begin{aligned}
||w_m - w_n||^2 &= ||(v - w_n) - (v - w_m)||^2 \\
&= -||(v - w_n) + (v - w_m)||^2 + 2(||v - w_n||^2 + ||v - w_m||^2) \\
&= -4||v - \underbrace{\tfrac{1}{2}(w_m + w_n)}_{\in W}||^2 + 2(||v - w_n||^2 + ||v - w_m||^2) \\
&\leq -4\delta^2 + 2(||v - w_n||^2 + ||v - w_m||^2) \\
&\underset{m,n\to\infty}{\to} -4\delta^2 + 2(\delta^2 + \delta^2) = 0.
\end{aligned}$$

Thus $\{w_n\}$ is a Cauchy sequence, and has a limit in $w_v$ in the Hilbert space $\mathcal{H}$. As $W$ is a closed subset of $\mathcal{H}$, $w_v \in W$. The norm is continuous, so $||v - w_v|| = \lim_{n\to\infty} ||v - w_n|| = \delta$.

For the second part, note that

$$||v - w_v||^2 = \delta^2 \leq ||v - w_v + \lambda w||^2, \qquad \forall \lambda \in \mathbb{C}, w \in W.$$

So

$$\begin{aligned}
0 &\leq ||v - w_v + \lambda w||^2 - ||v - w_v||^2 \\
&= \lambda\langle v - w_v | w\rangle + \bar{\lambda}\langle w | v - w_v\rangle + |\lambda|^2||w||^2, \qquad \forall \lambda \in \mathbb{C}, w \in W.
\end{aligned}$$

In particular, fix $w$ and allow for $\lambda = \mu\langle w | v - w_v\rangle$ with real $\mu$, so

$$0 \leq 2\mu|\langle v - w_v | w\rangle|^2 + \mu^2|\langle v - w_v | w\rangle|^2||w||^2, \qquad \forall \mu \in \mathbb{R}.$$

For small negative $\mu$, the first term dominates, but it is non-positive, so we must have $\langle v - w_v | w\rangle = 0$. Since $w \in W$ is arbitrary, $v - w_v \in W^\perp$.

The uniqueness parts are omitted. □

**Theorem 5.2.** *Let $W$ be a* closed *subspace of a Hilbert space $\mathcal{H}$. Then $\mathcal{H} = W \oplus W^\perp$.*

*Proof.* Lemma 5.1 shows that $\mathcal{H} = W + W^\perp$, and the latter is a direct sum since $W \cap W^\perp = \{0\}$. □

Theorem 5.2 says that orthogonal decomposition, (5.1), holds in Hilbert spaces. In this sense, Hilbert spaces share some geometric features with Euclidean spaces.

*Exercise* 5.2. Let $W$ be a subspace of a Hilbert space $\mathcal{H}$.

- Check that $W^\perp$ is a closed subspace.
- Show that $W^\perp = \overline{W}^\perp$.
- Use Lemma 5.1 to deduce that

$$\overline{W} = W^{\perp\perp}.$$

*Exercise* 5.3. Check that $c_{00}$ is a dense subspace of $\ell^2$, but it is not closed. Similarly for the continuous functions $C[0,1]$ inside $L^2[0,1]$.

## 5.2 Orthonormal bases

**Definition 6.** Let $\mathcal{H}$ be a Hilbert space. A family $\{e_i\}_{i\in I}$ is *orthonormal* if

$$\langle e_i | e_j \rangle = \delta_{ij}, \qquad \forall\, i, j, \in I.$$

It is called an *orthonormal basis* for $\mathcal{H}$ if the closure of the linear span of the $e_i$ is $\mathcal{H}$.

Here, "linear span" is an algebraic notion — only finite linear combinations are allowed. When we take the closure of the linear span, convergent infinite linear combinations are allowed. So the linear span of an orthonormal basis is *dense* in the Hilbert space.

**Theorem 5.3.** *Every Hilbert space has an orthonormal basis, and the cardinalities of any two such bases are the same. A separable Hilbert space has a countable orthonormal basis.*

Here, a separable space is one which has a countable dense subset. *We will assume separability of Hilbert spaces unless otherwise stated.*

*Proof.* This basically follows from a Zorn's Lemma argument (details omitted). □

**Theorem 5.4** (Parseval–Plancherel)**.** *Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for a (separable, infinite-dimensional) Hilbert space $\mathcal{H}$. The map*

$$\begin{aligned} f : \mathcal{H} &\to \ell^2 \\ \psi &\mapsto (\langle e_i | \psi \rangle)_{i\in\mathbb{N}} \end{aligned}$$

*is an isomorphism of Hilbert spaces. If $\varphi, \psi \in \mathcal{H}$, their inner product may be written as*

$$\langle\varphi|\psi\rangle = \sum_{i\in\mathbb{N}} \langle\varphi|e_i\rangle\langle e_i|\psi\rangle. \tag{5.2}$$

*In particular,*

$$||\psi||^2 = \sum_{i\in\mathbb{N}} |\langle e_i|\psi\rangle|^2.$$

*Proof.* Exercise. □

Eq. (5.2) in Theorem 5.4 is symbolically written in physics as the insertion of a "resolution of identity", or "completeness relation",

$$\langle\varphi|\psi\rangle = \langle\varphi| \left( \sum_{i\in\mathbb{N}} |e_i\rangle\langle e_i| \right) |\psi\rangle, \qquad \left( \sum_{i\in\mathbb{N}} |e_i\rangle\langle e_i| \right) = 1_{\mathcal{H}}.$$

The precise meaning of the right side is actually not so straightforward, since it involves an infinite sum of rank-1 projection *operators*, whose convergence requires an understanding of operator topologies.

## 5.3 Duality

If $V$ is a normed vector space (over $\mathbb{C}$), the space of *continuous* linear functionals $V \to \mathbb{C}$ is called the *dual space*, and is denoted $V^*$. We may define a norm on $V^*$ via

$$||f|| := \sup_{||v||=1} |f(v)|, \qquad f \in V^*.$$

In fact, $V^*$ becomes a Banach space in this way (Exercise).

In particular, for Hilbert spaces, each vector $\varphi \in \mathcal{H}$ is associated to a linear functional,

$$\langle\varphi| : \psi \mapsto \langle\varphi|\psi\rangle, \qquad \psi \in \mathcal{H}, \tag{5.3}$$

and $\langle\varphi|$ is continuous (why?). Remarkably, the converse is true as well; more precisely, there is the following duality between $\mathcal{H}$ and $\mathcal{H}^*$.

**Theorem 5.5** (Riesz representation theorem)**.** *The assignment*

$$\begin{aligned} g : \mathcal{H} &\to \mathcal{H}^* \\ \varphi &\mapsto \langle\varphi| \end{aligned}$$

*is antilinear, norm-preserving, and bijective. Thus $\mathcal{H}^*$ is a Hilbert space under the inner product*

$$\langle f|f'\rangle_{\mathcal{H}^*} = \langle g^{-1}(f')|g^{-1}(f)\rangle_{\mathcal{H}}$$

*and it is canonically anti-isomorphic to $\mathcal{H}$.*

*Proof.* Let $f \neq 0 \in \mathcal{H}^*$. Since $\ker(f)$ is a proper closed subspace of $\mathcal{H}$, Theorem 5.2 says that $\ker(f)^\perp$ is non-zero, and we may pick some nonzero $\varphi \in \ker(f)^\perp$. We shall verify that

$$\tilde{\varphi} = \frac{\overline{f(\varphi)}}{||\varphi||^2}\varphi$$

represents $f$. To see this, first observe that given any $\psi \in \mathcal{H}$, we have

$$\psi - \tfrac{f(\psi)}{f(\varphi)}\varphi \in \ker(f) \;\Rightarrow\; \langle\varphi|\psi - \tfrac{f(\psi)}{f(\varphi)}\varphi\rangle = 0.$$

It follows that

$$\begin{aligned}\langle\tilde{\varphi}|\psi\rangle = \frac{f(\varphi)}{||f(\varphi)||^2}\langle\varphi|\psi\rangle &= \frac{f(\varphi)}{||f(\varphi)||^2}\langle\varphi|\psi - \tfrac{f(\psi)}{f(\varphi)}\varphi + \tfrac{f(\psi)}{f(\varphi)}\varphi\rangle \\ &= \frac{f(\varphi)}{||f(\varphi)||^2}\langle\varphi|\tfrac{f(\psi)}{f(\varphi)}\varphi\rangle = f(\psi) \qquad\qquad \forall\psi \in \mathcal{H},\end{aligned}$$

and therefore $f = \langle\tilde{\varphi}|$. That this representative is unique is straightforward. The remaining statements are left as an exercise. □

*Remark.* The notation $\langle\varphi|$ for the linear function (5.3) is due to Dirac, who called it a "bra vector". He also wrote $|\psi\rangle \equiv \psi$ and called it a "ket vetor". The result of applying $\langle\varphi|$ to a vector $\psi$ is the numerical "bra-ket" $\langle\varphi|\psi\rangle \in \mathbb{C}$.

*Remark* 2. It is useful to think of $\psi = |\psi\rangle \in \mathcal{H}$ as the linear map

$$|\psi\rangle : \mathbb{C} \to \mathcal{H}, \qquad \lambda \mapsto \lambda\psi,$$

and compare it to the associated linear functional

$$\langle\psi| : \mathcal{H} \to \mathbb{C}.$$

Indeed, we shall see that the duality between $|\psi\rangle$ and $\langle\psi|$ is a special case of the adjoint operation.

## 5.4 Bounded/continuous linear operators

**Definition 7.** Let $V_1, V_2$ be normed vector spaces. A linear operator $T : V_1 \to V_2$ is *bounded*, or *continuous*, if its *operator norm* is finite,

$$||T||_{\text{op}} := \sup_{||\psi||=1} ||T\psi|| < \infty.$$

We write $\mathcal{B}(V_1, V_2)$ for the normed linear space of bounded linear operators $V_1 \to V_2$. If $V_1 = V_2 = V$, we simply write $\mathcal{B}(V)$.

**Notation:** By default, $||T||$ refers to the operator norm above.

*Exercise* 5.4. Check that the boundedness condition in Definition 7 is equivalent to continuity. Also check that $|| \cdot ||_{\text{op}}$ is indeed a norm, and that it is submultiplicative,

$$||T \circ T'||_{\text{op}} \leq ||T||_{\text{op}} \, ||T'||_{\text{op}}, \qquad T' \in \mathcal{B}(V_1, V_2), T \in \mathcal{B}(V_2, V_3).$$

*Exercise* 5.5. Let $V_1$ be a normed vector space and let $V_2$ be a Banach space. Suppose $V_0 \subset V_1$ is a dense linear subspace, and $T : V_0 \to V_2$ is a linear operator such that

$$||T||_{\text{op}} = \sup_{v \in V_0 : ||v||=1} ||Tv|| < \infty.$$

Then there is a unique extension of $T$ to an operator $\tilde{T} \in \mathcal{B}(V_1, V_2)$, and $||\tilde{T}||_{\text{op}} = ||T||_{\text{op}}$.

For example, a Hilbert space operator $\mathcal{H}_1 \to \mathcal{H}_2$ is often initially defined by its action on some orthonormal basis of $\mathcal{H}_1$. Then one takes $V_0$ to be the linear span of that orthonormal basis, and applies Exercise 5.5 to get a bounded operator $\mathcal{H}_1 \to \mathcal{H}_2$.

*Exercise* 5.6. Check that if $V_2$ is a Banach space, then $\mathcal{B}(V_1, V_2)$ with the operator norm is a Banach space. (In particular, this applies to the space of bounded operators $V \to V$ on a Banach/Hilbert space $V$.)

## 5.5 Adjoints of Hilbert space operators

We specialize to Hilbert spaces, and recall the bra-ket duality (Remark 2). There is a corresponding duality at the level of Hilbert space operators.

**Definition 8.** Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. The *adjoint* of $T$ is defined to be the operator $T^* \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ satisfying

$$\langle T^*\varphi|\psi\rangle = \langle\varphi|T\psi\rangle, \qquad \forall\, \varphi \in \mathcal{H}_2, \psi \in \mathcal{H}_1. \tag{5.4}$$

The existence and uniqueness of an adjoint operator is not completely obvious. One way to show this is via the following correspondence result:

**Lemma 5.6.** *If $T \in \mathcal{B}(\mathcal{H})$, then*

$$f_T : \mathcal{H} \oplus \mathcal{H} \to \mathbb{C}, \qquad (\varphi, \psi) \mapsto \langle\varphi|T\psi\rangle$$

*is a bounded sesquilinear form, whose norm*

$$||f_T|| := \sup_{||\varphi||,||\psi||=1} |f_T(\varphi, \psi)|$$

*equals $||T||_{\text{op}}$. Conversely, if $f$ is a bounded sesquilinear form, there exists a unique $T_f \in \mathcal{B}(\mathcal{H})$, such that*

$$f(\varphi, \psi) = \langle\varphi|T_f\psi\rangle \qquad \forall\varphi, \psi \in \mathcal{H}.$$

*Proof.* Exercise. The Riesz representation theorem is needed here. □

*Remark.* If you know about weak operator topologies, try to figure out how a bounded operator is approximated by finite linear combinations of "ket-bra" operators,

$$\begin{aligned} |\psi\rangle\langle\varphi| : \mathcal{H} &\to \mathcal{H} \\ \xi &\mapsto \langle\varphi|\xi\rangle\psi. \end{aligned}$$

The adjoint of $|\psi\rangle\langle\varphi|$ is easily checked to be $|\varphi\rangle\langle\psi|$. With this in mind, the adjoint duality for Hilbert space operators is a natural generalization of the Riesz duality for Hilbert spaces and their duals.

*Exercise* 5.7. For operators $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $T' \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$, show that

- $T \mapsto T^*$ is antilinear.
- $(T^*)^* = T$.
- $||T^*|| = ||T||$.

- $(T'T)^* = T^*T'^*$.
- If $T$ is invertible, then so is $T^*$, and $(T^*)^{-1} = (T^{-1})^*$.
- $||T^*T|| = ||TT^*|| = ||T||^2 = ||T^*||^2$.

As mentioned in Remark 2, we can think of $|\psi\rangle$ as a linear map $\mathbb{C} \to \mathcal{H}$ with adjoint $\langle\psi| : \mathcal{H} \to \mathbb{C}$. The kernel of $\langle\psi|$ comprises those vectors which are orthogonal to $\psi$, i.e.,

$$\ker(\langle\psi|) = (\mathrm{Ran}|\psi\rangle)^\perp.$$

More generally, the adjoint of $T \in \mathcal{B}(\mathcal{H})$ has the same property,

$$\ker(T^*) = (\mathrm{Ran}\, T)^\perp. \tag{5.5}$$

This follows from

$$T^*\psi = 0 \Leftrightarrow \langle\varphi|T^*\psi\rangle = 0 \Leftrightarrow \langle T\varphi|\psi\rangle = 0, \qquad \varphi, \psi \in \mathcal{H}.$$