# 11 Tangent bundle and vector bundles

## 11.1 Cotangent vectors and gradient

Recall the definitions of velocities of curves, tangent vectors, and tangent spaces in (2.4).

*Exercise* 11.1. Let $v \in T_xX$ be an arbitrary derivation at $x$. Fix any coordinate chart around $x$, and write $x^i : U \to \mathbb{R}$ for the $i$-th coordinate function. Show that $v$ is a linear combination,

$$v = \sum_{i=1}^{d} v^i \partial_i|_x, \tag{11.1}$$

where $v^i := v(x^i) \in \mathbb{R}$. Also show that $\{\partial_i|_x\}_{i=1,\dots,d}$ is a linearly independent set of derivations at $x$. This shows that $T_xX \cong \mathbb{R}^d$ (non-canonically).

*Exercise* 11.2. Let $v \in T_xX$. Show that it arises as the velocity vector of the following curve,

$$\gamma : (-\epsilon, \epsilon) \to U \subset X, \qquad t \mapsto \varphi^{-1}(tv^1, \dots, tv^n),$$

where $v^i$ are the numerical components of $v$ as in (11.1).

*Exercise* 11.3. Let $(U, \varphi)$ and $(\tilde{U}, \tilde{\varphi})$ be two charts, $x^i, \tilde{x}^j$ be their respective coordinate functions, and $\partial_i|_x, \tilde{\partial}_j|_x$ the respective coordinate tangent vectors, $i, j = 1, \dots, d$. For $x \in U \cap \tilde{U}$, show that

$$\partial_i|_x = \frac{\partial \tilde{x}^j}{\partial x^i}\Big|_{\varphi(x)} \tilde{\partial}_j|_x. \qquad \text{(summation convention)}$$

Deduce that the components of $v \in T_xX$ transform as

$$\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}\Big|_{\varphi(x)} v^i. \tag{11.2}$$

Notice that $\left(\frac{\partial \tilde{x}^j}{\partial x^i}\Big|_{\varphi(x)}\right)_{ji}$ is precisely the Jacobian matrix of first partial derivatives of the change-of-coordinates map $\tilde{\varphi} \circ \varphi^{-1}$. Above, and subsequently, we adopt the *Einstein summation convention* of summing over repeated indices when they appear both as a superscript and as a subscript.

Recall the directional derivative of $f$ along $\gamma$,

$$\frac{d(f\circ\gamma)}{dt}\Big|_{t=0} = \dot\gamma(0)(f). \tag{11.3}$$

Fixing $\gamma$ and varying $f$, this is a derivation (denoted $\dot\gamma$). Fixing the function $f : X \to \mathbb{R}$, (11.3) depends linearly on $\dot\gamma(0)$. So we have a linear functional,

$$\begin{aligned} df_x &: T_xX \to \mathbb{R} \\ &\dot\gamma(0) \mapsto \dot\gamma(0)(f) \equiv \frac{d(f\circ\gamma)}{dt}\Big|_{t=0}. \end{aligned} \tag{11.4}$$

This $df_x$ is the coordinate-independent version of $\mathrm{grad}\check f|_{\varphi(x)}$.

**Definition 32.** Let $X$ be a manifold. The *cotangent space at* $x \in X$, denoted $T^*_xX$, is the space of linear functionals $T_xX \to \mathbb{R}$, and its elements are called *cotangent vectors at* $x$.

*Exercise* 11.4. Let $x^i : U \to \mathbb{R}, i = 1,\dots,d$ be local coordinate functions for a chart around $x$. Show that $(dx^i)_x, i = 1,\dots,d$ are linearly independent cotangent vectors at $x$, dual to $\partial_i|_x, i-1,\dots,d$.

The map $df_x : T_xX \to \mathbb{R}$ of Eq. (11.4) is sometimes said to be the "linearization" of a function $f : X \to \mathbb{R}$ at $x$. Actually, it is better to think of the codomain of $f$ as the affine *manifold* $\mathbb{R}$ with $\mathbb{R}$-action, and $df_x$ as a linear map

$$df_x : T_xX \to \underbrace{T_{f(x)}\mathbb{R}}_{=\mathbb{R}}.$$

Above, the linear isomorphism $T_{f(x)}\mathbb{R} = \mathbb{R}$ is the canonical isomorphism (2.8). Then Eq. (11.4) becomes a special case of the following concept.

**Definition 33.** Let $f : X \to Y$ be a smooth map. Its *derivative at* $x \in X$ is the linear map

$$df_x : T_xX \to T_{f(x)}Y$$

defined by

$$df_x(v)(g) := v(g\circ f), \qquad v \in T_xX,\ g \in C^\infty(Y). \tag{11.5}$$

*Exercise* 11.5. Check the following:

- $df_x(v)$ is indeed a derivation of $C^\infty(Y)$ at $f(x)$.
- $df_x$ is indeed a linear map.
- The chain rule is satisfied: if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a composition of smooth maps, then $d(g \circ f)_x = dg_{f(x)} \circ df_x$.
- $d(\mathrm{id}_X)_x = \mathrm{id}_{T_xX}$.
- If $f : X \to Y$ is a diffeomorphism, then $df_x$ is a linear isomorphism whose inverse is $d(f^{-1})_{f(x)}$.

Let us describe $df_x$ with respect to charts $(U, \varphi)$ and $(V, \psi)$ around $x$ and $f(x)$, respectively. Write $x^i$ and $\tilde{x}^i$ for the respective coordinates. The coordinate representation of $f$ is

$$\check{f} := \psi \circ f \circ \varphi^{-1} = (\check{f}^1, \ldots, \check{f}^m), \qquad m = \dim Y.$$

Going through the definitions, we have

$$\begin{aligned} df_x\left(\partial_i|_x\right)(g) &\equiv \left(\partial_i|_x\right)(g \circ f) \\ &= \frac{\partial (g \circ f \circ \varphi^{-1})}{\partial x^i}\Big|_{\varphi(x)} \\ &= \frac{\partial (g \circ \psi^{-1})}{\partial \tilde{x}^j}\Big|_{\psi(f(x))} \frac{\partial (\psi \circ f \circ \varphi^{-1})^j}{\partial x^i}\Big|_{\varphi(x)} \\ &= \left(\left(\tilde{\partial}_j|_{f(x)}\right)(g)\right) \cdot \frac{\partial \check{f}^j}{\partial x^i}\Big|_{\varphi(x)} \\ &= \left(\frac{\partial \check{f}^j}{\partial x^i}\Big|_{\varphi(x)} \left(\tilde{\partial}_j|_{f(x)}\right)\right)(g). \qquad g \in C^\infty(Y) \end{aligned}$$

By linearity of $df_x$, for a general $v = v^i \partial_i|_x$, we have

$$df_x(v) = v^i \frac{\partial \check{f}^j}{\partial x^i}\Big|_{\varphi(x)} \left(\tilde{\partial}_j|_{f(x)}\right).$$

In other words, with respect to the coordinate tangent bases, the linear map $df_x$ is represented as the Jacobian matrix of $\check{f}$,

$$df_x \longleftrightarrow \left(\frac{\partial \check{f}^j}{\partial x^i}\Big|_{\varphi(x)}\right)_{ji} =: J(\check{f})|_{\varphi(x)}. \tag{11.6}$$

Put in another way, $df_x$ is the coordinate-independent way to refer to the Jacobian at $x$. There is a simple geometric way to describe what $df_x$ does:

**Proposition 11.1.** *Let $f : X \to Y$ be a smooth map, and $\gamma$ be a curve in $X$. Then*

$$df_x(\dot{\gamma}(0)) = (f \dot{\circ} \gamma)(0).$$

*Proof.* Exercise.

□

## 11.2 Submanifolds and transversality (Recap)

In the following, we consider $k \leq d$ and write $\mathbb{R}^k \subset \mathbb{R}^d$ for the standard slice with last $d - k$ coordinates being zero.

**Definition 34.** A $k$-dimensional *embedded/regular submanifold* (or simply *submanifold*) of a $d$-manifold $X$ is a subspace $Z \subset X$ such that every point $z \in Z$ is contained in a chart $(U, \varphi)$ of $X$ with the property that $\varphi(U \cap Z) = \varphi(U) \cap \mathbb{R}^k$.

A submanifold $Z \subset X$ is a manifold in its own right. Specifically, we take the charts $\varphi|_{U \cap Z} : U \cap Z \to \mathbb{R}^k$, and check that they make $Z$ into a $k$-manifold. For example, any open subset of a manifold $X$ is a submanifold.

*Exercise* 11.6. Let $f : X \to Y$ be a smooth map. Show that if $X'$ is a submanifold of $X$, then $f|_{X'}$ remains smooth. Similarly, show that if $Y'$ is a submanifold of $Y$ such that $f(X) \subset Y'$, then $f : X \to Y'$ remains smooth.

**Definition 35.** A smooth map $f : X \to Y$ is a *immersion* (resp. *submersion*) *at* $x \in X$, if $df_x$ is injective (resp. surjective). If $f$ is an immersion at every $x \in X$, it is simply called an *immersion*; likewise for submersions.

*Remark* 4. The following facts may be found in standard texts on smooth manifolds or differential topology, e.g. the books of Guillemin–Pollack or Lee.

- If $df_x$ is an isomorphism, then the *Inverse Function Theorem* says that $f : X \to Y$ is actually a *local diffeomorphism* at $x$, i.e., there is an open neighbourhood $U$ of $x$ such that $f : U \to f(U)$ is a diffeomorphism.
- Write $\dim X = d, \dim Y = m$. If $f$ is an immersion at $x$, then in some coordinate charts centred at $x$ and $f(x)$, it is represented as a standard immersion $(x^1, \ldots, x^d) \mapsto (x^1, \ldots, x^d, 0, \ldots, 0)$ of $\mathbb{R}^d$ into $\mathbb{R}^m$.

- Similarly, if $f$ is a submersion at $x$, then it has a coordinate representation as a standard projection $(x^1, \ldots, x^m, x^{m+1}, \ldots, x^d) \mapsto (x^1, \ldots, x^m)$ of $\mathbb{R}^d$ onto $\mathbb{R}^m$.

**Lemma 11.2.** *Let $\pi : E \to X$ be a surjective submersion. Then a map $f : X \to Y$ is smooth iff $f \circ \pi : E \to Y$ is smooth.*

*Proof.* This follows from Remark 4 (Exercise).

□

**Definition 36.** If $f : X \to Y$ is a smooth map, then $y \in Y$ is a *regular value* of $f$ if $f$ is a submersion at all points in the (possibly empty) preimage $f^{-1}(y)$.

**Proposition 11.3.** *Let $y$ be a regular value of a smooth map $f : X \to Y$. Then $f^{-1}(y)$ is a submanifold of $X$ with codimension equal to* $\dim Y$.

*Proof.* Exercise; use Remark 4. □

In applications, manifolds often arise as submanifolds of $\mathbb{R}^n$, by virtue of being level sets $f^{-1}(y)$ of some function $f : \mathbb{R}^n \to \mathbb{R}^m$, for regular values $y$. For example, the unit spheres $S^{n-1} \subset \mathbb{R}^n$ arise this way. It is much easier to use Prop. 11.3, than to manually construct an atlas for $f^{-1}(y)$.

An important generalization of regularity is the notion of transversality.

**Definition 37.** A submanifold $Z \subset Y$ is *transverse* to $f : X \to Y$ if

$$\text{Image}(df_x) + T_{f(x)}(Z) = T_{f(x)}Y, \qquad \forall\, x \in f^{-1}(Z).$$

**Theorem 11.4.** *Let $Z \subset Y$ be a submanifold transverse to a map $f : X \to Y$. Then $f^{-1}(Z)$ is a submanifold of $X$, with*

$$\operatorname{codim} f^{-1}(Z) = \operatorname{codim} Z.$$

*Proof.* See §5 of Guillemin–Pollack or Theorem 6.30 of Lee. □

The following corollary will be of relevance to us:

**Corollary 11.5.** *Let $\pi : E \to X$ be a surjective submersion. Then for any submanifold $Z \subset X$, the preimage $\pi^{-1}(Z)$ is a submanifold of $E$.*

*Proof.* The stated assumptions on $\pi$ implies that it is transverse to any submanifold $Z \subset X$, whence the conclusion follows Theorem 11.4. $\square$

In the context of Corollary 11.5, if $\pi : E \to X$ is a surjective submersion, we have the commutative diagram of smooth maps,

$$\begin{array}{ccc} \pi^{-1}(Z) & \xrightarrow{\tilde{\iota}} & E \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\iota} & X \end{array},$$

with $\iota, \tilde{\iota}$ being inclusions of submanifolds. Informally, for each $x \in X$, we think of $\pi^{-1}\{x\}$ as the "fibre" above $x$, and $E$ as a collection of such fibres, "smoothly" parametrized by the base manifold $X$. This collection has a localizability property: we can restrict to smooth subcollections of fibres lying over smaller base submanifolds $Z \subset X$. So far, this is almost like the notion of smooth "fibre bundle" that we will need for physics applications. What is missing is whether/how the various fibres $\pi^{-1}\{x\}$ are identifiable with each other.

## 11.3 (Co)tangent bundle

As a set, the *tangent bundle* of a manifold $X$ is a disjoint union

$$TX = \bigsqcup_{x \in X} T_x X,$$

with a surjective projection map

$$\pi : TX \to X, \qquad (x, v_x) \mapsto x.$$

Importantly, $TX$ is regarded as a $2d$-manifold, and the projection map $\pi : TX \to X$ is smooth. This is valid because:

- $X$ is a $d$-manifold so it has smooth atlas $(U_\alpha, \varphi_\alpha), \alpha \in \mathcal{I}$.
- Over each $U_\alpha$, consider the "local tangent bundle",

$$TU_\alpha := \pi^{-1}(U_\alpha) = \bigsqcup_{x \in U_\alpha} T_x X.$$

For any $x \in U_\alpha$ and $v_x \in T_xX$, we have the expansion $v_x = v_x^i \partial_i|_x$ in terms of the $\alpha$-coordinate tangent vectors. So there are bijections

$$\tilde{\varphi}_\alpha : TU_\alpha \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^d \subset \mathbb{R}^{2d}$$
$$(x, v_x) \mapsto (\varphi_\alpha(x); v_x^1, \ldots, v_x^d)$$

onto open subsets of $\mathbb{R}^{2d}$.

- Topologize $TX$ by taking a basis of open sets to be

$$\tilde{\varphi}_\alpha^{-1}(V), \qquad V \text{ open in } \mathbb{R}^{2d}, \alpha \in \mathcal{I}.$$

  (The Hausdorff and second-countability properties should be checked.)

- The $TU_\alpha, \alpha \in \mathcal{I}$ provide an open cover of $TX$, and the $\tilde{\varphi}_\alpha, \alpha \in \mathcal{I}$ become charts of an atlas making $TX$ a topological $2d$-manifold.

- The change-of-coordinates maps $\tilde{\varphi}_\alpha \circ \tilde{\varphi}_\beta^{-1}$ are smooth. This is because:

  - The base change-of-coordinates map $\varphi_\alpha \circ \varphi_\beta^{-1}$ are smooth since $X$ is a manifold.
  - The vector coordinates $v^i$ transform according to the smooth formula Eq. (11.2).

  Thus, $\{TU_\alpha, \tilde{\varphi}_\alpha\}_{\alpha \in \mathcal{I}}$, form a smooth atlas for $TX$, making it a $(2d)$-manifold. (Pass to the maximal atlas as usual.)

- The $\{TU_\alpha, \tilde{\varphi}_\alpha\}_{\alpha \in \mathcal{I}}$ form a special atlas, called a *bundle atlas*. If we use the local chart $(TU_\alpha, \tilde{\varphi}_\alpha)$ for $TX$ and $(U_\alpha, \varphi_\alpha)$ for $X$, the local description of the projection map $\pi : TX \to X$ is simply the map $\varphi_\alpha(U_\alpha) \times \mathbb{R}^d \to \varphi_\alpha(U_\alpha)$ projecting onto the first factor. The fact that $\tilde{\varphi}_\alpha$ respects the projection-to-base structure, says that $\varphi_\alpha(U_\alpha)$ provides a local trivialization (see Definition 38, Example 11.1).

  In particular, $\pi : TX \to X$ is a smooth surjective map. In fact, it is a submersion as well (why?).

Similarly, for a coordinate chart $(U, \varphi)$, the contangent spaces $T_x^*X, x \in U$ can be spanned by $\{dx^i|_x\}_{i=1,\ldots,d}$ (Exercise 11.4). So $\omega_x \in T_x^*X$ is a linear combination $\omega_x = \omega_{x,i} dx^i|_x$. We may construct the *cotangent bundle*

$$T^*X = \bigsqcup_{x \in X} T_x^*X, \qquad \pi : T^*X \to X, \qquad (x, \omega_x) \mapsto x,$$

in the same manner as the tangent bundle.

*Exercise* 11.7. Let $\omega \in T^*_x X$. Show that its components with respect to $\{(dx^i)_x\}_{i=1,\dots,d}$ are

$$\omega_{x,i} = \omega_x\big(\partial_i|_x\big).$$

Check that under change-of-coordinates $x^i \to \tilde{x}^j$, the $\omega_{x,i}$ transform as

$$\omega_{x,i} = \frac{\partial \tilde{x}^j}{\partial x^i}\Big|_{\varphi(x)} \tilde{\omega}_{x,j} \qquad \text{or} \qquad \tilde{\omega}_{x,j} = \frac{\partial x^i}{\partial \tilde{x}^j}\Big|_{\psi(x)} \omega_{x,i}. \tag{11.7}$$

## 11.4 General vector bundles

**Definition 38.** A (real) *vector bundle of rank* $n$ is a smooth surjective map $\pi : E \to X$, called the *bundle projection*, such that there exists an open cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ of $X$ and diffeomorphisms,

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n, \qquad \alpha \in \mathcal{I},$$

called *local trivializations*, satisfying

- Write $\pi_\alpha : U_\alpha \times \mathbb{R}^n \to U_\alpha$ for the projection map. Then

$$\pi_\alpha\big(\Phi_\alpha(p)\big) = \pi(p), \qquad \forall\, p \in \pi^{-1}(U_\alpha). \tag{11.8}$$

- For any pair of local trivializations $(U_\alpha, \Phi_\alpha), (U_\beta, \Phi_\beta)$, we have

$$\Phi_\alpha \circ \Phi_\beta^{-1}(x, \mathbf{v}) = (x, g_{\alpha\beta}(x) \cdot \mathbf{v}), \qquad x \in U_\alpha \cap U_\beta, \mathbf{v} \in \mathbb{R}^n, \tag{11.9}$$

  for smooth *transition functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$.

*Complex* vector bundles are defined in the same way, except that $\mathbb{R}^n$ is replaced by $\mathbb{C}^n$, and $g_{\alpha\beta}$ is $\mathrm{GL}(n, \mathbb{C})$-valued. A vector bundle of rank $n = 1$ is also called a *line bundle*.

*Example* 11.1. The tangent bundle $\pi : TX \to X$ of a $d$-manifold is a basic example of a vector bundle of rank $n = d$. To see this, recall that we have the charts $(TU_\alpha, \tilde{\varphi}_\alpha)$ with covering $TX$. Then

$$\Phi_\alpha : \underbrace{TU_\alpha}_{=\pi^{-1}(U_\alpha)} \xrightarrow{\tilde{\varphi}_\alpha} \varphi_\alpha(U_\alpha) \times \mathbb{R}^d \xrightarrow{\varphi_\alpha^{-1}\times \mathrm{id}_{\mathbb{R}^d}} U_\alpha \times \mathbb{R}^d$$

are diffeomorphisms respecting the projections maps $\pi$ and $\pi_\alpha$ (Eq. (11.8)). So these $\Phi_\alpha$ are local trivializations. By (11.2), the transition functions are

$$g_{\alpha\beta} : x \mapsto \Big( \frac{\partial x^j_\alpha}{\partial x^i_\beta}\Big|_{\varphi_\beta(x)} \Big)_{ji}, \qquad x \in U_\alpha \cap U_\beta,$$

which are smooth and $\mathrm{GL}(d, \mathbb{R})$-valued.

Similarly, the cotangent bundle $\pi : T^*X \to X$ is a vector bundle of rank $n = d$. Comparing (11.7) to (11.2), we deduce that the transition functions of $T^*X$ are the transpose-inverse of the transition functions of $TX$.

*Exercise* 11.8. Show that the projection map of a vector bundle is a (surjective) submersion. (Follow the case of tangent bundles.)

**Terminology:** A vector bundle has the *availability* of a bundle atlas of local trivializations $\{U_\alpha, \Phi_\alpha\}_{\alpha\in\mathcal{I}}$. Sometimes, we specify a vector bundle by giving a convenient initial bundle atlas of local trivializations. But we really allow for all possible compatible local trivializations (in the sense of Eq. (11.9)). So, formally, a vector bundle is equipped with a maximal bundle atlas of mutually compatible local trivializations. (Compare with the situation for manifolds.)

For the cartesian product $X \times \mathbb{K}^n$, we say that $\pi : X \times \mathbb{K}^n \to X$ is a *trivial* vector bundle. Because there exist local trivializations $\Phi_\alpha$, a general vector bundle $\pi : E \to X$ is said to be *locally trivializable*, i.e., locally identifiable with a trivial vector bundle $\pi_\alpha : U_\alpha \times \mathbb{K}^n \to U_\alpha$. If in the (maximal) bundle atlas, there exists some local trivialization $\Phi_\alpha$ with $U_\alpha = X$, then $\Phi_\alpha$ is called a *(global) trivialization* of $E$, and $E$ is said to be *(globally) trivializable*. Generally speaking, even when some (global) trivialization is available, we do not pick any preferred one, so $E$ is not canonically identified with a trivial vector bundle.

## 11.5 Sections of vector bundles

**Definition 39.** A *section* of a vector bundle $\pi : E \to X$ is a smooth map $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$. The space of sections of $E$ is denoted $\Gamma(E)$. If $s$ is only defined on an open subset $U \subset X$, it is called a *local section* over $U$.

*Remark.* For each $x \in X$, the fibre $E_x := \pi^{-1}(x)$ inherits a vector space structure through its identification with $\mathbb{K}^n$ via any local trivialization.

Under pointwise addition and scalar multiplication, $\Gamma(E)$ is also a vector space. Actually, we can pointwise multiply $s \in \Gamma(E)$ by $f \in C^\infty(X)$. So $\Gamma(E)$ is a module over $C^\infty(X)$.

### 11.5.1 Tangent vector fields

**Definition 40.** A *tangent vector field* over $X$ is a section $v : X \to TX$. We also write $\mathfrak{X}(X) := \Gamma(TX)$ for the space of tangent vector fields.

*Example* 11.2. We can also consider sections of a local tangent bundle $TU$ (where $U \subset X$ is the domain of some coordinate chart). For example, the map $\partial_i : U \to TU$ taking $x \mapsto \partial_i|_x$ is a local tangent vector field, defined over $U$.

*Exercise* 11.9. Let $v : X \to TX$ be a map such that $\pi \circ v = \mathrm{id}_X$, but with $v$ not assumed to be smooth. Pointwise in $X$, we have the derivation $v_x$ of $C^\infty(X)$ at $x$. In total, we have a function

$$\begin{aligned} v(f) : X &\to \mathbb{R} \\ x &\mapsto v_x(f), \qquad f \in C^\infty(X). \end{aligned} \tag{11.10}$$

Check that the following are equivalent:

1. For all $f \in C^\infty(X)$, the function $v(f)$ defined by Eq. (11.10) lies in $C^\infty(X)$.
2. $v$ is a (smooth) tangent vector field.

Exercise 11.9 verifies that a (smooth) tangent vector field $v$ defines a *derivation* of the algebra $C^\infty(X)$, i.e.,

- $v : C^\infty(X) \to C^\infty(X)$, $f \mapsto v(f)$ is linear;
- $v(f \cdot g) = f \cdot v(g) + v(f) \cdot g, \qquad f, g \in C^\infty(X)$.

Conversely, if $D$ is a derivation on $C^\infty(X)$, then we have pointwise derivations $D_x \in T_x X$ at each $x$ (i.e., tangent vectors at $x$) given by the formula

$$D_x(f) := (Df)(x), \qquad f \in C^\infty(X).$$

The assignment $x \mapsto D_x$ is a (smooth) tangent vector field, due to Exercise 11.9.

### 11.5.2 Differential 1-forms

**Definition 41.** A *(differential) 1-form* on a manifold $X$ is a section of $T^*X$. We also write $\Omega^1(X) := \Gamma(T^*X)$ for the space of 1-forms on $X$.

A 1-form $\omega \in \Omega^1(X)$ pairs with a tangent vector field $v \in \mathfrak{X}(X)$, to give a smooth $\mathbb{R}$-valued function,

$$\omega(v) \in C^\infty(X), \qquad \omega(v)(x) := \omega_x(v_x). \tag{11.11}$$

*Exercise* 11.10. Show that the map (11.11) is $C^\infty(X)$-linear,

$$\omega(f \cdot v) = f \cdot \omega(v), \qquad f \in C^\infty(X),\ v \in \mathfrak{X}(X).$$

Conversely, show that any $C^\infty(X)$-linear map $\mathfrak{X}(X) \to C^\infty(X)$ must be represented by a 1-form according to (11.11).

*Example* 11.3. Let $f \in C^\infty(X)$. At each $x$, we have the gradient cotangent vector (Eq. (11.4)),

$$\begin{aligned} df_x : T_xX &\to T_{f(x)}\mathbb{R} = \mathbb{R} \\ v_x &\mapsto v_x(f). \end{aligned}$$

Altogether, we have the *gradient 1-form* of $f$,

$$df \in \Omega^1(X), \qquad (df)(x) = df_x \in T^*_xX.$$

*Example* 11.4. We also consider local sections of $T^*X$. For example, the *coordinate 1-forms* $(dx^i), i = 1, \dots, d$, are sections of $T^*U = (T^*X)|_U$, and they are dual to the coordinate tangent vector fields $\partial_i$.

*Example* 11.5. Cover the unit circle $S^1$ with two charts $U, \tilde{U}$, with respective angular coordinates $\theta \in (0, 2\pi)$ and $\tilde{\theta} \in (-\pi, \pi)$. The intersection $U \cap \tilde{U}$ is the union of the open upper half-circle and the open lower half-circle. The change-of-coordinates map, $\theta \mapsto \tilde{\theta}$, is the identity map on the upper half-circle, and the shift-by-$(-2\pi)$ map on the lower half-circle. We have *local* coordinate 1-forms $d\theta$ and $d\tilde{\theta}$, which coincide over $U \cap \tilde{U}$ (why?). So they describe a single globally defined 1-form. It is customary to just write $d\theta$ for this global 1-form, even though the coordinate function $\theta$ is only locally defined.

### 11.5.3 Frames and local trivializations

Recall from Remark 11.5 that each fibre $E_x$ of a vector bundle $\pi : E \to X$ is a vector space, thus it has a zero element. Therefore, it makes invariant sense to ask whether a section $s \in \Gamma(E)$ vanishes at a given basepoint $x$.

A general vector bundle $E$ may not admit any nowhere-vanishing section. But a trivial vector bundle $X \times \mathbb{K}^n$ does — for $i = 1, \ldots, n$ take

$$s_i : x \mapsto (x, e_i),$$

where $\{e_i\}_{i=1,\ldots,n}$ is the standard basis for $\mathbb{K}^n$. In fact, the nowhere-vanishing sections $s_1, \ldots, s_n$ provide, at each $x \in X$, a basis for the fibre $E_x$.

**Definition 42.** Let $\pi : E \to X$ be a rank-$n$ vector bundle over $X$, and $U \subset X$ be an open subset. An $n$-tuple of local sections $(s_1, \ldots, s_n)$ is called a *local frame for $E$ over $U$*, if $\{s_i(x)\}_{i=1,\ldots,n}$ is a basis for $E_x$ for each $x \in U$.

A local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{K}^n$ determines a local frame for $E$ over $U$: set $s_i(x) = \Phi^{-1}(x, e_i)$. Then over $U$, an arbitrary section $s \in \Gamma(E)$ is expanded as

$$\Phi(s(x)) = v^i(x) s_i(x), \qquad x \in U,$$

for smooth *local component functions* $v^i : U \to \mathbb{K}$.

Conversely:

*Exercise* 11.11. Let $(s_1, \ldots, s_n)$ be a local frame for $E$ over $U$. Define the map

$$\Psi : U \times \mathbb{K}^n \to \pi^{-1}(U)$$
$$(x, (v^1, \ldots, v^n)) \mapsto \sum_{i=1}^{n} v^i s_i(x).$$

Check that $\Psi$ is a diffeomorphism, making $\Psi^{-1}$ a local trivialization of $E$.

## 11.6 Morphisms of vector bundles

**Definition 43.** A *morphism of vector bundles* from $\pi : E \to X$ to $\pi' : E' \to X'$ is a smooth map $F : E \to E'$, such that there exists a map $f : X \to X'$ making the following diagram commute

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{f} & X' \end{array}, \tag{11.12}$$

and such that the induced fibrewise maps

$$F|_{E_x} : E_x \to E'_{f(x)}, \qquad x \in X,$$

are linear.

*Remark.* The map $f$ making (11.12) commute is uniquely determined by $F$, and it is actually automatically smooth, due to Lemma 11.2.

*Example* 11.6. Let $Z \subset X$ be a submanifold, and consider $E|_Z := \pi^{-1}(Z)$. By Corollary 11.5, $\pi : E|_Z \to Z$ is still a vector bundle. The inclusion map $F : E|_Z \to E$ is a vector bundle morphism.

### 11.6.1 Derivative as tangent bundle morphism

We know that a smooth map $f : X \to Y$ linearizes, at each $x \in X$, to a linear map $df_x : T_xX \to T_{f(x)}Y$ (Eq. (11.5)). Assembling all these linear maps, we get the *(total) derivative* of $f$,

$$df : TX \to TY,$$
$$df = \bigsqcup_{x\in X} df_x : \bigsqcup_{x\in X} T_xX \to \bigsqcup_{x\in X} T_{f(x)}Y.$$

Since we know that $TX$ and $TY$ are smooth vector bundles over $X$ and $Y$, it makes sense to verify that $df$ is a vector bundle morphism in the sense of Definition 43. The algebraic conditions are satisfied by, so the only thing to check is that $df$ is a smooth map.

Let $\check{f} = \psi \circ f \circ \phi^{-1}$ be the representation of $f$ with respect to some charts $(U, \varphi)$ and $(V, \psi)$ around $x$ and $f(x)$ respectively. Recall from Eq. (11.6) that $df_x$ is represented as the Jacobian matrix $J(\check{f})|_{\varphi(x)}$. Therefore, in the corresponding local trivializations $(TU, \Phi)$ and $(TV, \Psi)$ for the tangent bundles, $df$ is represented as

$$\Psi \circ df \circ \Phi^{-1} : (\underbrace{x^1, \ldots, x^n}_{\mathbf{x}}; \underbrace{v^1 \ldots, v^n}_{\mathbf{v}}) \mapsto (\check{f}(\mathbf{x}); J(\check{f})|_{\mathbf{x}}(\mathbf{v})),$$

which is smooth, since $\check{f}$ and $J(\check{f})$ are smooth. Thus $df$ is smooth.

It follows from Exercise 11.5 that the derivative is *functorial*:

- Chain Rule: If $f : X \to Y$ and $g : Y \to Z$ are smooth maps, then $d(g \circ f) = dg \circ df : TX \to TZ$;

- $d(\mathrm{id}_X) = \mathrm{id}_{TX}$.

In particular, if $f : X \to Y$ is a diffeomorphism, then $df : TX \to TY$ is a vector bundle *isomorphism*, with inverse morphism $(df)^{-1} = d(f^{-1})$.

## 11.7 Tautological line bundle over $\mathbb{RP}^1$

The *real projective line* $\mathbb{RP}^1$ is the set of one-dimensional subspaces $\ell$ in $\mathbb{R}^2$. On $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, define the equivalence relation $(x, y) \sim (x', y')$ iff $(x, y)$ and $(x', y')$ lie in the same line. So $\mathbb{RP}^1$ is the quotient space (with quotient topology) under this equivalence relation,

$$q : \mathbb{R}^2 \setminus \{0\} \to \mathbb{RP}^1, \qquad (x, y) \mapsto [(x, y)].$$

(Thus $U \subset \mathbb{RP}^1$ is open iff $q^{-1}(U)$ is open in $\mathbb{R}^2 \setminus \{0\}$.)

Let $\ell_{\mathrm{H}} = [(1, 0)]$ and $\ell_{\mathrm{V}} = [(0, 1)]$, and define

$$U_1 = \mathbb{RP}^1 \setminus \{\ell_{\mathrm{V}}\}, \qquad U_2 = \mathbb{RP}^1 \setminus \{\ell_{\mathrm{H}}\}.$$

Then $\{U_1, U_2\}$ is an open cover of $\mathbb{RP}^1$. Define the maps

$$\begin{aligned} \varphi_1 : U_1 \to \mathbb{R}, &\qquad [(x, y)] \mapsto y/x, \\ \varphi_2 : U_2 \to \mathbb{R}, &\qquad [(x, y)] \mapsto x/y, \end{aligned}$$

which are well-defined homeomorphisms (why?). On $U_1 \cap U_2 = \mathbb{RP}^1 \setminus \{\ell_{\mathrm{V}}, \ell_{\mathrm{H}}\}$, we have $\varphi_i(U_1 \cap U_2) = \mathbb{R} \setminus \{0\}$ for $i = 1, 2$. The change-of-coordinates map is the smooth map

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}, \qquad \lambda \mapsto \lambda^{-1},$$

thus $\mathbb{RP}^1$ is a smooth 1-manifold.

The *tautological line bundle* $\pi : \mathcal{L}^{\mathbb{R}} \to \mathbb{RP}^1$ is the following subset (with subspace topology) of $\mathbb{RP}^1 \times \mathbb{R}^2$,

$$\mathcal{L}^{\mathbb{R}} = \{(\ell; (x, y)) \in \mathbb{RP}^1 \times \mathbb{R}^2 \,|\, (x, y) \in \ell\},$$

with $\pi : (\ell; (x, y)) \mapsto \ell$. Define the maps

$$\begin{aligned} \Phi_1 : \pi^{-1}(U_1) \to U_1 \times \mathbb{R}, &\qquad (\ell; (x, y)) \mapsto \left(\ell; x\sqrt{1 + (\varphi_1(\ell))^2}\right), \\ \Phi_2 : \pi^{-1}(U_2) \to U_2 \times \mathbb{R}, &\qquad (\ell; (x, y)) \mapsto \left(\ell; y\sqrt{1 + (\varphi_2(\ell))^2}\right), \end{aligned}$$

with inverses

$$\Phi_i^{-1} : U_i \times \mathbb{R} \to \pi^{-1}(U_i) \subset \mathcal{L}^{\mathbb{R}} \subset \mathbb{RP}^1 \times \mathbb{R}^2$$
$$(\ell; \mu) \mapsto \begin{cases} \left(\ell; \mu\left(\frac{1}{\sqrt{1+(\varphi_1(\ell))^2}}, \frac{\varphi_1(\ell)}{\sqrt{1+(\varphi_1(\ell))^2}}\right)\right), & i = 1, \\ \left(\ell; \mu\left(\frac{\varphi_2(\ell)}{\sqrt{1+(\varphi_2(\ell))^2}}, \frac{1}{\sqrt{1+(\varphi_2(\ell))^2}}\right)\right), & i = 2. \end{cases}$$

These formulae show that $\Phi_i$ are homeomorphisms. Then

$$(\varphi_i \times \mathrm{id}_{\mathbb{R}}) \circ \Phi_i : \pi^{-1}(U_i) \to \mathbb{R} \times \mathbb{R}, \qquad i = 1, 2,$$

provide an atlas of local coordinate charts for $\mathcal{L}^{\mathbb{R}}$ as a topological manifold. The change-of-coordinates map is

$$(\varphi_2 \times \mathrm{id}_{\mathbb{R}}) \circ \Phi_2 \circ \Phi_1^{-1} \circ (\varphi_1 \times \mathrm{id}_{\mathbb{R}}),$$

which is smooth, due to (exercise)

$$\Phi_2 \circ \Phi_1^{-1} : (U_1 \cap U_2) \times \mathbb{R} \to (U_1 \cap U_2) \times \mathbb{R}$$
$$(\ell; \mu) \mapsto \big(\ell; \mathrm{sgn}(\varphi_2(\ell))\mu\big), \tag{11.13}$$

being a smooth map (why?). This shows that $\mathcal{L}^{\mathbb{R}}$ a smooth 2-manifold.

Using the above local coordinates, we see that $\pi : \mathcal{L}^{\mathbb{R}} \to \mathbb{RP}^1$ is smooth, the $\Phi_i$ are diffeomorphisms, and $\pi_i \circ \Phi_i = \pi$. Furthermore, Eq. (11.13) says that the transition function is

$$g_{21} = \mathrm{sgn} \circ \varphi_2 : U_1 \cap U_2 \to \{\pm 1\} = \mathrm{O}(1) \subset \mathrm{GL}(1, \mathbb{R}),$$

which is smooth (why?). Thus $\pi : \mathcal{L}^{\mathbb{R}} \to \mathbb{RP}^1$ is a real line bundle.

*Exercise* 11.12. Show that $\pi : \mathcal{L}^{\mathbb{R}} \to \mathbb{RP}^1$ does not admit any nowhere-vanishing section. (Thus it is not trivializable).

Exercise 11.12 shows that any smooth section of $\pi : \mathcal{L}^{\mathbb{R}} \to \mathbb{RP}^1$ must encounter some zeroes. This demonstrates that $\Gamma(\mathcal{L}^{\mathbb{R}})$ and $C^\infty(\mathbb{RP}^1)$ are very different spaces. For example, they are inequivalent modules over $C^\infty(\mathbb{RP}^1)$ (the latter is free but the former is not).

*Remark.* For each $\ell \in \mathbb{RP}^1$, the fibre elements are vectors $v \in \ell \subset \mathbb{R}^2$. We can give them the norm inherited from $\mathbb{R}^2$. The local trivializations $\Phi_i$ of $\mathcal{L}^{\mathbb{R}}$ preserve the norm. For example, if $\ell \in U_1$, then

$$(x, y) \in \ell \;\Rightarrow\; |x\sqrt{1 + (\varphi_1(\ell))^2}|^2 = |x\sqrt{1 + (y/x)^2}|^2 = x^2 + y^2 = ||(x, y)||^2.$$

Correspondingly, the transition function $\Phi_2 \circ \Phi_1^{-1}$ is O(1)-valued.