Large-scale quantization and coarse cohomology

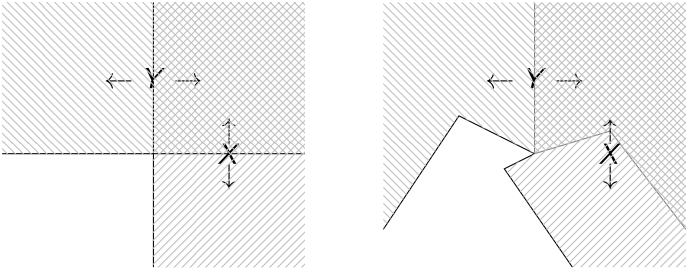
Guo Chuan Thiang 程国传
BICMR, Peking University

ACJSU II — NCG and K-theory — Tokyo

1 June 2023

Main result (with M. Ludewig arXiv:2306.?????)

M metric space, $P = P^2$ any rapid decrease projection on $L^2(M)$.



$$\boxed{\mathrm{Tr}[P\chi_X P, P\chi_Y P] \in \tfrac{i}{2\pi}\cdot\mathbb{Z}}$$

Physics "proof": Quantized to 10^{-9} accuracy!

Maths: $\langle$ Coarse partition ; K-theory $\rangle$.
Measures delocalization of P.

1D partitioned/odd coarse index (sketch)

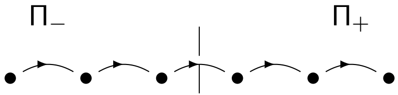
- World's most famous commutator:

$$[\text{Position}, \underbrace{\text{Momentum}}_{-i\frac{d}{dx}=D}] = i.$$

- Cayley transform $U = \exp(\pi i \chi(D))$.

$$\mathrm{Tr}(U^*[\Pi_+, U]) = \mathrm{Tr}(U^*\Pi_+ U - \Pi_+) \overset{\text{why??}}{=} 1.$$

Discrete shift S:



$$\mathrm{Tr}(S^* \underbrace{[\Pi_+, S]}_{\text{finite rank}}) = \mathrm{Tr}(S^*\Pi_+ S - \Pi_+) \overset{\text{why??}}{\in} \mathbb{Z}.$$

I. Story-telling and Motivation

- **Hall, 1879**: 2D sample M subject to $\perp$ magnetic field. *Transverse* response of charges to electric field in M:

$$\sigma_{\mathrm{Hall}} \in \mathbb{R}.$$

- **Weyl, 1929**: Gauge principle (in quantum mechanics).
- **Landau, 1930**: QM of magnetic Laplacian[1] on $\mathbb{R}^2$: Spectrum "quantizes" to a set of infinitely-degenerate Landau levels.

$$\bullet\, b \qquad \bullet\, 3b \qquad \bullet\, 5b \qquad \bullet\, 7b \qquad \cdots$$

- **Schrödinger, 1932**:

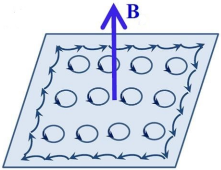
$$\frac{1}{\sqrt{g}} \nabla_k \sqrt{g}\, g^{kl} \nabla_l - \frac{R}{4} - \frac{1}{2} f_{kl}\, s^{kl} = \mu^2$$

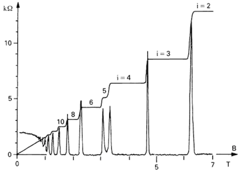
[1] $\nabla_b^* \nabla_b = -(\partial_x^2 + \partial_y^2) + 2ibx\partial_y + b^2 x^2$

Physics $\to$ Maths

- **Early 80s surprise**: QHE at low temperatures, large field,

$$\boxed{\sigma_{\mathrm{Hall}} \in \mathbb{Z}.}$$



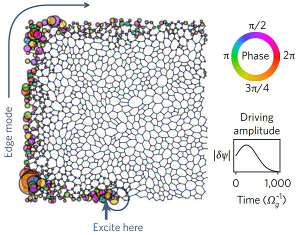


- **90s**: Bellissard+: NCG, cyclic theory for NC-2-torus.
- **Ludewig+Kubota+T, 2020s**:

$$[P_{\mathrm{Landau}}] = [\text{Dirac coarse index}].$$

- **Today**: Coarse index $\leftrightsquigarrow$ quantization of trace (thus σ_{Hall}).

“Small-scale structure” irrelevant! QHE works on bumpy samples.

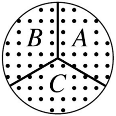


N. Mitchell et al, Nature Phys. (2018)

2000s: Coarse geometry was anticipated (A. Kitaev)

Anyons in an exactly solved model and beyond

Alexei Kitaev *



$$\nu(P) = h(A,B,C) \stackrel{\text{def}}{=} \sum_{j\in A}\sum_{k\in B}\sum_{l\in C} h_{jkl}.$$

Periodic table for topological insulators and superconductors

Alexei Kitaev

Theorem: *Any gapped local free-fermion Hamiltonian in $\mathbb{R}^d$ is equivalent to a texture.*

(That is the key technical result, but I cannot explain it in any detail in such a short note.) Discrete systems on a compact metric space L are classified by the K-homology group $K_q^{\mathbb{R}}(L)$.

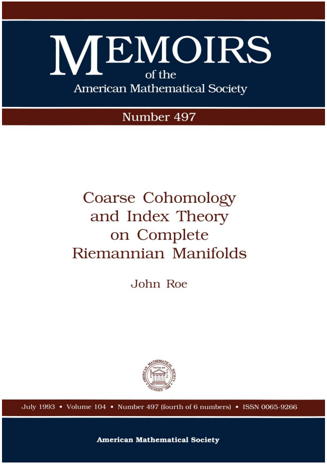
30. N. Higson, and J. Roe, *Analytic K-homology*, Oxford University Press, New York, 2000.
31. A. Connes, *Noncommutative geometry*, Academic Press, San Diego, 1994.

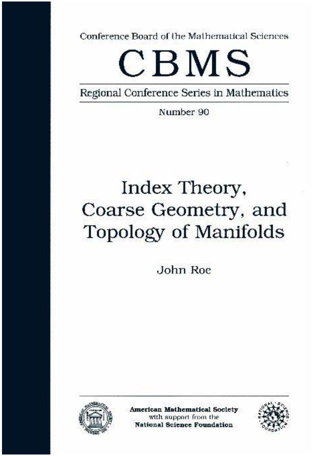
In general, a *quasidiagonal matrix* is a lattice-indexed matrix $A = (A_{jk})$ with sufficiently rapidly decaying off-diagonal elements. Technically, one requires that

$$|A_{jk}| \leqslant c|j-k|^{-\alpha}, \quad \alpha > d,$$

where c and α are some constants, and d is the dimension of the space. Note that "lattice" is simply a way to impose coarse $\mathbb{R}^d$ geometry at large distances. We may think about the problem in these terms: matrices are operators acting in some Hilbert space, and lattice points are basis vectors. But the choice of the basis need not be fixed. One may safely replace the basis vector corresponding to a given lattice point by a linear combination of nearby points. One may also use some kind of coarse-graining, replacing the basis by a decomposition into orthogonal subspaces corresponding to groups of points, or regions in $\mathbb{R}^d$.

Coarse cohomology and geometry





Finite propagation operators $\rightsquigarrow$ **Roe C^*-algebras**.
"Middle ground" needed...

- Maths: Properties of *points–space*. Coarse index **obstructions** (e.g. no psc metrics as corollary).
- Physics: *Operators–wavefunctions*. Coarse index counts something $\rightsquigarrow$ **large-scale spectral phenomena**!

Dirac operators coupled to vector potentials

(elliptic operators/index theory/characteristic classes/anomalies/gauge fields)

M. F. Atiyah† and I. M. Singer‡

- Twisted Diracs exhibit coarse/higher index explicitly[2]!
- Quantization involves *gauge*: $(\text{Planck})/(\text{electron charge})^2$.

[2] For $\mathbb{Z}^d$, higher index is families index over moduli of $B\mathbb{Z}^d$ ("T-duality").

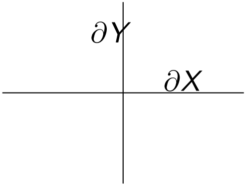
II: Coarse cohomology

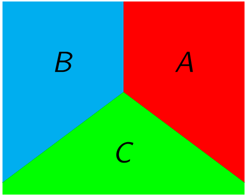
- q-cochains are maps $\varphi : M^{q+1} \to \mathbb{Z}$ with usual δ-operator.

$$\delta(f_0 \otimes f_1) = 1 \otimes f_0 \otimes f_1 - f_0 \otimes 1 \otimes f_1 + f_0 \otimes f_1 \otimes 1.$$

- **Coarse** 0-cochain = compactly-supported f.
- Generally, $\varphi = f_0 \otimes \ldots \otimes f_q$ is **coarse** if:

$$\bigcap_{i=0}^{q} \mathrm{Pen}(\mathrm{Supp}(f_i); R) \quad \text{is compact,} \qquad \forall R \geq 0.$$





- Antisymmetrized coarse complex $\to HX^\bullet(M)$.
- **Partitions** $A \sqcup B \sqcup C = M$ define non-trivial cohomology classes!

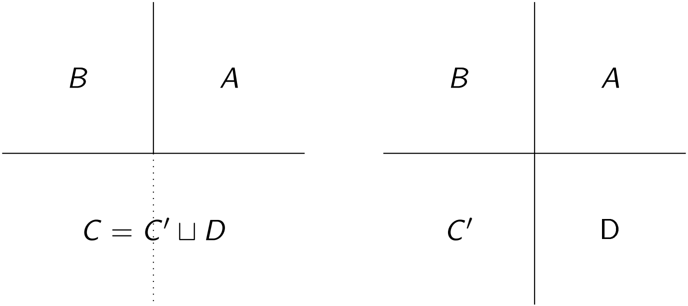
$$\mathrm{Pen}(A; R) \cap \mathrm{Pen}(B; R) \cap \mathrm{Pen}(C; R) \quad \text{compact} \ \ \forall R > 0.$$

$$\varphi_{A,B,C} := \chi_A \otimes \chi_B \otimes \chi_C + \text{antisymm.}$$

- Compare Schick–Zadeh multi-partitioned manifold index theorem.

How does $[\varphi_{A,B,C}]$ interact with operators on M?

Cobordism invariance



$$[\varphi_{A,B,C}] = \begin{cases} [\varphi_{B,C'\sqcup D,A}], \\ [\varphi_{A,B\sqcup C',D}], \\ [\varphi_{D,\underbrace{A\sqcup B}_{Y},C'}], \\ [\varphi_{C',\underbrace{D\sqcup A}_{X},B}], \end{cases} \quad \in HX^2(M).$$

- $L_0, \ldots, L_q \in \mathscr{B}_{\mathrm{fin}}(M)$ locally trace class[3] and **finite propagation**:

$$\langle f_0 \otimes \ldots \otimes f_q; L_0, \ldots, L_q \rangle := \mathrm{Tr}\Big((f_0 L) \ldots (f_q L_q)\Big) < \infty.$$

- Coboundary–**Projection** trivializes,

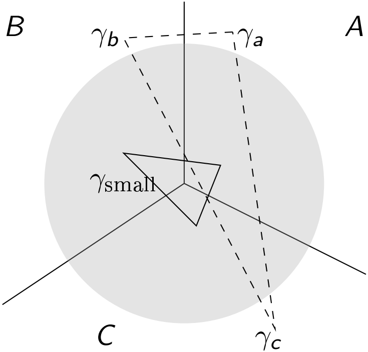
$$\langle \delta\varphi; P \rangle \equiv \langle \delta\varphi; \underbrace{P, \ldots, P}_{q+1} \rangle = 0, \qquad P = P^2 \in \mathscr{B}_{\mathrm{fin}}(M), \ q \text{ **even**}.$$

- Partition-projection pairing:

$$\begin{aligned} \langle A, B, C; P \rangle &:= \langle [\varphi_{A,B,C}]; P \rangle, \\ &= \mathrm{Tr}\big(\chi_A P \chi_B P \chi_C P + \mathrm{antisymm}\big). \end{aligned}$$

[3] On $L^2(M)$; fL and Lf are trace class whenever f has compact support.

$$\begin{aligned}\langle A, B, C; P\rangle &\equiv \mathrm{Tr}\big(\chi_A P \chi_B P \chi_C P + \text{antisymm}\big)\\ &= \ldots = \mathrm{Tr}[P\chi_A P, P\chi_B P].\end{aligned}$$



Commutator-trace equals to "sum over loop-amplitudes":

$$\left(\sum_{\text{anticlockwise }\gamma} - \sum_{\text{clockwise }\gamma}\right) P(\gamma_a, \gamma_b) P(\gamma_b, \gamma_c) P(\gamma_c, \gamma_a)$$

Cobordism invariance leads to another formulation:

$$2\langle A, B, C; P\rangle = \mathrm{Tr}[P\chi_X P, P\chi_Y P] \overset{\text{Kubo}}{\equiv} \frac{i}{2\pi} \cdot \sigma_{\mathrm{Hall}}(P)$$

Quantum response along ∂Y when electric field applied along ∂X.

Experimentally: this trace is

- **Finite** for a large class of ∞-dimensional P.
- **Integer** multiple of a universal constant.
- Stable against perturbations in M-**geometry** and/or P.

Rigorous explanation only known[4] for $M = \mathbb{R}^2$ or $M = \Gamma = \mathbb{Z}^2$.
Real sample curvature variation $\gg 10^{-9}$...

[4] Hyperbolic: CHMMM+T

The pairing is **zero** when

- P is finite rank/trace class.
- $P\chi_X P$ or $P\chi_Y P$ is trace class (any X, Y).
- $P\chi_X P\chi_Y P$ is trace class. (Lidskii)
- P has **finite propagation**.
- $P = \overline{P}$ (need gauge-connection!)

Only "fully delocalized" part of P can contribute.
"Localized noisy states" filtered out — plateaux[5]

Need infinite propagation P to get "integer$\neq$ 0".

Roe C^*-algebras not suitable for $\mathrm{Tr}(\cdot)$ ...

[5] Exact rounding off was the surprise that led to Nobel prize.

III: Fréchet algebra of rapid decrease operators[6]

[6] Polynomial volume growth of M assumed.

Definition: $\mathscr{B}(M)$ is space of bounded operators on $L^2(M)$ with finite "decay seminorms" for all $\nu \geq 0$:

$$\rho_\nu(L) := \sup_{\mathrm{rad}(V),\mathrm{rad}(W)\leq 1} ||\underbrace{\chi_V L \chi_W}_{\substack{\text{matrix}\\ \text{elements}}}||_{\mathrm{Tr}}(1 + d(V, W))^\nu < \infty$$

For each $Z \subset M$, define $\mathscr{B}(M; Z) \subset \mathscr{B}(M)$ as subspace with

$$\rho_{\nu,Z}(L) := \sup_{\mathrm{rad}(V),\mathrm{rad}(W)\leq 1} ||\chi_V L \chi_W||_{\mathrm{Tr}} \big(1+d(V, Z)\big)^\nu \big(1+d(W, Z)\big)^\nu < \infty$$

- Automatically locally trace class: $\mathscr{B}_{\mathrm{fin}}(M) \subset \mathscr{B}(M)$.
- $\mathscr{B}(M; Z)$ not *closed* ideal in $\mathscr{B}(M)$.
  (Different topologies for different Z!)
- Algebra property not obvious.
- "Natural" seminorms are not *submultiplicative*.
- Nevertheless, we establish that $\mathscr{B}(M; Z)$ are m-**convex Fréchet algebras**, so they have their own holomorphic functional calculi.

- If $Z \subset M$ is compact, then

$$\mathscr{B}(M; Z) \subset \{\text{trace class}\}.$$

- If $Z_0, \dots, Z_q \subset M$ are **poly-coarsely transverse**,

$$\bigcap_{i=0}^{q} \mathrm{Pen}(Z_i\,; r) \subset \mathrm{Pen}\left(\bigcap_{i=0}^{q} Z_i\,; R(r)\right), \qquad \forall\, r > 0,$$

with R at most **polynomial** in r, then

$$\mathscr{B}(M; Z_0) \cdot \ldots \cdot \mathscr{B}(M; Z_q) = \mathscr{B}\Big(M; \bigcap_{i=0}^{q} Z_i\Big),$$

with continuous multiplication.

IV: Proof of integrality

For any $P = P^2 \in \mathscr{B}(M)$, and $\partial X, \partial Y$ poly-coarsely transverse, write $P_X := P\chi_X P$ and $P_Y := P\chi_Y P$. We want to prove:

$$\boxed{\mathrm{Tr}[P_X, P_Y] \in \frac{i}{2\pi} \cdot \mathbb{Z}}$$

(Also want cobordism invariance, homotopy invariance, promote to K-theory.)

Clearly: P_X is X-localized, and P_Y is Y-localized.

What about $[P_X, P_Y]$?

- P_X fails to remain a projection by an error near ∂X:

$$P_X - P_X^2 = P \underbrace{\chi_X P(1-\chi_X)}_{\partial_X\text{-localized}} P \in \mathscr{B}(M;\partial X).$$

- Holomorphic calculus leads to

$$e^{2\pi i z} - 1 = z(z-1)f(z)$$
$$e^{2\pi i P_X} - 1 = \underbrace{P_X(1-P_X)}_{\mathscr{B}(M;\partial X)} \underbrace{f(P_X)}_{\mathscr{B}(M)^+} \in \mathscr{B}(M;\partial X).$$

- Similarly for Y,

$$e^{2\pi i P_Y} - 1 \in \mathscr{B}(M;\partial Y).$$

- Our localization theorems now apply,

$$(\underbrace{e^{2\pi i P_X} - 1}_{\partial X}) \cdot (\underbrace{e^{2\pi i P_Y} - 1}_{\partial Y}) \in \mathscr{B}(M; \underbrace{\partial X \cap \partial Y}_{\text{compact}}) \subset \text{trace class}.$$

- Kitaev's conjecture: this trivializes the Fredholm determinant,

$$\det\left(e^{2\pi i P_X} e^{2\pi i P_Y} e^{-2\pi i P_X} e^{-2\pi i P_Y}\right) = 1.$$

- By Pincus–Helton–Howe (BCH formula), above is equivalent to

$$\exp\left(\mathrm{Tr}[2\pi i\, P_X, 2\pi i\, P_Y]\right) = 1 \quad \Rightarrow \mathrm{Tr}[P_X, P_Y] \in \tfrac{i}{2\pi}\mathbb{Z}.$$

If S, T are invertible and

$$(S-1)(T-1), \quad (T-1)(S-1), \quad (S^*-1)(T-1), \quad (T-1)(S^*-1)$$

are trace class, then $\det(STS^{-1}T^{-1}) = 1$.

- Our work provides a big natural class of examples:

$$(e^{2\pi i P_X} - 1)(e^{2\pi i P_Y} - 1) \text{ is trace class, etc.}$$

  for any RD projection P on $L^2(M)$, and any poly-coarsely transverse axes $\partial X, \partial Y$.
- Then $\mathrm{Tr}[P_X, P_Y]$ is quantized!

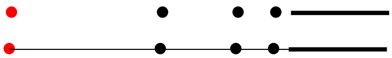
- Need to couple Dirac/Schrödinger to large gauge field to obtain non-trivial P.
- Basic example: Landau-level spectral projection P.

$$[P_{\mathrm{Landau}}] = [\ker(D_b)] = \mathrm{coarse\text{-}ind}(\mathrm{Dirac}) \neq 0 \in K_0(C^*(M)).$$

- Coarse index obstructs existence of localized basis for $\mathrm{Range}(P)$. (L+T, '22).
- Coarse-MV principle implies spectral-gap filling theorem[7] for the magnetic Laplacian restricted to any $\approx$ half-space X.

Spec on M

Spec on X



[7] Ludewig+Kubota+T, CMP '21, '22.

- Working with $\mathscr{B}(M)$ rather than C^*-algebras is crucial for quantized trace formula for P.
- For K-theory: Is $\mathscr{B}(M)$ spectral in $C^*(M)$?
  Actually, suffices that $[P] \neq 0$ in $K_0(C^*(M))$ (Baum–Connes).
- $HX^\bullet_{\mathrm{poly}}$ different from standard one?
- Deliberately avoided cyclic cohomology: KO-torsion?
  Extension to unknown operator space...
- Full coarse invariance?
- $\dim > 2$?