Quantum kilogram and index theory on noncompact Riemannian manifolds

Guo Chuan Thiang

Beijing International Center for Mathematical Research, Peking University

December 4, 2023

Abstract

Details for talks at BICMR and Fudan. Largely based on ideas in arXiv:2308.02819. Geometry, gauge theory, and index theory of Landau levels are explained. An explicit "elementary" calculation of a numerical index for Landau levels is provided. The role of large-scale index theory in forcing macroscopic quantization of Hall conductance is explained. A connection to remarkable formulae by Helton–Howe on commutator-traces is mentioned. The role of the quantized Hall conductance in allowing universal access to Planck's constant, leading to the modern quantum definition of the kilogram, is discussed.

1 Commutators, traces and QM

Essence of QM:

$$[\text{Position}, \text{Momentum}] = [x, -i\nabla] = i. \qquad (\times\hbar) \tag{1}$$

Need ∞-dim Hilbert space $\mathcal{H}$: taking trace gives contradiction.

- A bounded operator A on $\mathcal{H}$ is *trace-class* if

$$\sum_k \langle e_k| \underbrace{|A|}_{\sqrt{A^*A}} e_k\rangle < \infty, \qquad \text{any ONB } \{e_k\}_{k\in\mathbb{N}},$$

  and its *trace* is defined as

$$\mathrm{Tr}(A) := \sum_k \langle e_k|Ae_k\rangle \in \mathbb{C}. \tag{2}$$

  These are *unitarily invariant* concepts.

- Trace-class is normed ideal *strictly* inside compacts; not C^*-algebra.

Quantum mechanics involves *mathematical framework* to predict *real numbers measured in experiments*. For example, the *operator* equality in Eq. (1) is not directly measurable, it must be turned into a number!

Furthermore, *symmetries*, in the sense of preserving observed quantities, are unitarily implemented (Wigner's theorem). In this regard, it is important that the trace, Eq. (2), is a unitary invariant.

Examples:

- "Mixed state" ρ in QM $\leftrightarrow$ positive operator with $\mathrm{Tr}(\rho)=1$. Expectation of any observable $T=T^*$ in state ρ is
$$\mathrm{Tr}(\underbrace{\rho T}_{\text{trace class}})=\mathrm{Tr}(T\rho)\in\mathbb{R}.$$
This is a rather strict notion of "state", in which any $T=T^*$ is "observable".
- Operator L on $L^2(M_{\mathrm{cpt}})$ with smooth integral kernel is trace-class. For general Riemannian manifold M, *local* truncations $\chi_K L\chi_{K'}$ by compact subsets K,K' are trace-class.

QM is very much about the failure of observables A,B to commute, so commutators $[A,B]$ are crucial.

- On $L^2(M)$, all multiplication operators by characteristic functions, $\chi_K, K\subset M$, commute.
- Informally, "location observables obviously commute". *Or do they...*?
- For example, let $X,Y\subset M=\mathbb{R}^2$ be right half-plane and upper half-plane respectively. The full Hilbert space $L^2(\mathbb{R}^2)$ doesn't distinguish clockwise/anticlockwise, so it shouldn't make a difference whether we look in X followed by Y (anticlockwise), or look in Y followed by X (clockwise).
- But Hilbert *subspaces* of $L^2(\mathbb{R}^2)$ *can* have distinguished orientation (see Bargmann space later). The compressions of X,Y to such a subspace will no longer be expected to commute! The x,y coordinates become "noncommutative". As we will see, the quantum Hall effect is, in some sense, a measurement of
$$[x\,\text{position}, y\,\text{position}]$$
within a "low energy" Hilbert subspace.

1.1 Trace of a commutator

If a commutator $[A,B]$ does manage to be trace-class, it seems that its trace must vanish,
$$\mathrm{Tr}[A,B]\stackrel{?}{=}\mathrm{Tr}(AB)-\mathrm{Tr}(BA)\stackrel{?}{=}0.$$
Actually, first equality is only valid when AB and BA are *individually* trace-class. A deep result of [Lidskii '59; Fredholm determinant],
$$\mathrm{Tr}(T)=\sum\text{eigenvalues}(T),$$
and equality of non-zero eigenvalues of AB and BA (including multiplicity), shows that the second equality holds.

We are interested the following situation:
$$\mathrm{Tr}[A,B]\text{“}{=}\text{”}\underbrace{\mathrm{Tr}(AB)}_{\infty}-\underbrace{\mathrm{Tr}(BA)}_{\infty}\in\mathbb{C}.$$

This should bring *index theory* ideas into mind, with a "coarse filter".

Example: Normalize position, momentum operators to have spectrum in unit interval $[-\frac{1}{2}, \frac{1}{2}]$:

$$\check{x} = f(x), \quad \check{p} = f(-i\nabla), \qquad f : t \mapsto \tfrac{t}{2\sqrt{1+t^2}}.$$

Explicit computation (cf. Elgart–Fraas '23):

$$2\pi i \cdot \mathrm{Tr}[\check{x}, \check{p}] = -1, \qquad (\times\hbar)$$

as numbers! (Contrast Eq. (1).)

2 General trace-quantization theorem

For a trace-class projection P, obviously $\mathrm{Tr}(P)$ is an integer, and this relates to ordinary Fredholm index.

Theorem 2.1 (cf. Ludewig–Thiang 2308:02819)**.** *Let C, D be bounded self-adjoint operators. Suppose*

$$[C, D] \qquad \text{trace class.}$$

If C, D are "almost projections" in the sense that

$$(C - C^2)(D - D^2) \quad \text{trace class,} \tag{3}$$

then the commutator-trace is quantized:

$$2\pi i \cdot \mathrm{Tr}[C, D] \in \mathbb{Z} \subset \mathbb{C}. \tag{4}$$

Proof. Experimental physics (later)! Mathematically:

- The holomorphic map $z \mapsto e^{2\pi i z} - 1$ has zeroes at $z = 0, 1$, so

  $$e^{2\pi i z} - 1 = \varphi(z) \cdot (z - z^2).$$

  Functional calculus:

  $$\big(e^{2\pi i C} - 1\big)\big(e^{2\pi i D} - 1\big) = \varphi(C) \cdot \underbrace{(C - C^2)(D - D^2)}_{\text{trace class by assumption (3)}} \cdot \varphi(D)$$

  is trace-class.

- Kitaev(formal, 2000)–Elgart–Fraas(rigorous, 2023): For unitaries U, V,

  $$(U - 1)(V - 1) \quad \text{trace class} \;\Rightarrow\; \det(UVU^{-1}V^{-1}) = 1.$$

  Apply this to $U = e^{2\pi i C}, V = e^{2\pi i D}$,

  $$\det\big(e^{2\pi i C} e^{2\pi i D} e^{-2\pi i C} e^{-2\pi i D}\big) = 1.$$

- For $[C, D]$ trace class, Pincus–HH identity (BCH) applies,

$$\exp\Big(\underbrace{\mathrm{Tr}[2\pi i C, 2\pi i D]}_{\Rightarrow \text{ in } 2\pi i\mathbb{Z}}\Big) = 1.$$

  and (4) follows immediately.

□

Example: In previous example, take

$$C = \hat{x} + \frac{1}{2}, \qquad D = \hat{p} + \frac{1}{2},$$

and check that $(C - C^2)(D - D^2)$ is trace class, e.g., using [Reed–Simon XI.21] to re-factorize into a product of two Hilbert–Schmidt operators. The use of our theorem will more generally illustrated by physical *families of Toeplitz-type operators*,

$$C = PXP \equiv P_X, \qquad D = PYP \equiv P_Y,$$

where P is some projection with "rapidly-decreasing" integral kernel, and X, Y are "partition operators". Generally, neither $P_X - P_X^2$ nor $P_Y - P_Y^2$ is Hilbert–Schmidt. Rather, the trace class condition on their product comes from geometric-analytic considerations.

2.1 Helton–Howe formula for Toeplitz operators on disc

Let $\mathbb{D} \subset \mathbb{C}$ be unit disc. The *Bergman space*

$$H^2(\mathbb{D}) \subset L^2(\mathbb{D})$$

comprises the holomorphics $\sum_{n=0}^{\infty} a_n z^n$, and is a Hilbert subspace.

- Smooth functions $f_1, f_2 \in C^\infty(\overline{\mathbb{D}})$ commute as multiplication operators on L^2.
- Let $P : L^2(\mathbb{D}) \to H^2(\mathbb{D})$ be the orthogonal projection. The compressions of f_i by P, denoted

$$P_{f_i} \equiv P f_i P,$$

  are called *Toeplitz operators*. They only commute up to trace-class, and the formula

$$2\pi i \cdot \mathrm{Tr}[P_{f_1}, P_{f_2}] = \int_{\mathbb{D}} df_1 \wedge df_2 = \int_{\mathbb{D}} \{f_1, f_2\}_{\text{P.B.}}$$

  was proven in [Helton–Howe, Acta '75].
- Similar formulae hold for Toeplitz operators on weighted higher-dimensional Bergman spaces [Tang–Wang–Zheng, Adv. Math. '23]. Trace-class membership and formulae need substantial work.

We are concerned with a special class of symbol functions — the (smoothened) characteristic functions. For subsets $A \subset \mathbb{D}$, we simply write A for the multiplication operator by χ_A.

Example:

Let X, Y be right half-space, and upper half-space, respectively. So dX, dY are bump 1-forms supported near vertical/horizontal axis respectively, while $dX \wedge dY$ is bump 2-form supported near origin, of mass 1. Then HH-formula gives

$$2\pi i \cdot \mathrm{Tr}[P_X, P_Y] = 1.$$

Furthermore, this *integer* is "cobordism invariant" with respect to X, Y, in the sense of only relying on the transversality of ∂X with ∂Y.

3 Quantum Hall effect

3.1 Magnetic fields

Laboratory $\sim$ Riemannian 3-manifold[1] $\mathcal{M}$ with metric g. A *magnetic field* over $\mathcal{M}$ is a closed *2-form* $\mathcal{F}$ on $\mathcal{M}$. When a point particle of electric charge q located at $x \in U$ is on a path with velocity vector $v \in T_x\mathcal{M}$, then it is accelerated by the *Lorentz force* vector $q(v \lrcorner \mathcal{F}_x)^\sharp$. Here, $\lrcorner$ denotes interior product, and $(\cdot)^\sharp$ is the conversion of a 1-form to a tangent vector field via the Riemannian metric g.

Confine the motion of charges to a 2-submanifold $M \subset \mathcal{M}$. This means that the component of forces normal to M is automatically counteracted, and only the tangential components are relevant for accelerating motion within M.

Write $\iota : M \hookrightarrow \mathcal{M}$ for the inclusion map. With velocity vector $v \in T_x M$, the tangential part of the Lorentz force 1-form at x is

$$\iota^*(v \lrcorner \mathcal{F}_x) = v \lrcorner \iota^*\mathcal{F}_x.$$

We see that only the data of $\iota^*\mathcal{F}$ is relevant for acceleration of motion within M.

Suppose M is orientable, then a choice of orientation gives a volume form vol_M. Since $\iota^*\mathcal{F}$ is a 2-form on M, it can be written as

$$\iota^*\mathcal{F} = B \cdot \mathrm{vol}_M \tag{5}$$

for some scalar function $B \in C^\infty(M)$. If B happens to be a constant function, we say that the magnetic field strength (as felt by a charged particle on M) is *uniform*.

3.1.1 Normal field strength

If $\mathcal{M}$ is orientable, then a choice of orientation turns $\mathcal{F}$ into the more familiar *magnetic vector field*

$$\mathbf{B} = (*\mathcal{F})^\sharp, \tag{6}$$

and vice versa. Here, $*$ is the Hodge star with respect to g and the orientation.

Remark 3.1. The vector field description of a magnetic field is not canonical — arrowheads on mangetic field lines reflect a choice of orientation[2].

Write $\mathcal{N} \to M$ for the normal line bundle over M, so

$$T\mathcal{M}|_M \equiv \iota^*T\mathcal{M} = TM \oplus \mathcal{N}.$$

If M and $\mathcal{M}$ are orientable, then so is $\mathcal{N}$, and $\mathcal{N}$ is trivializable. Choosing orientations on M and $\mathcal{M}$, we get an "outward-pointing" unit normal vector field $\mathbf{n}$ over M. So over M, the magnetic vector field $\mathbf{B}$ has a well-defined normal component function,

$$B^\perp = g(\mathbf{B}, \mathbf{n}) \in C^\infty(M).$$

[1]Typically, this is just taken to be an open subset of Euclidean $\mathbb{R}^3$.
[2]Classical electromagnetism does not violate parity by picking out a preferred (local) orientation.

In fact, $B^\perp$ coincides with B of Eq. (5); let $\{\mathbf{e}_1, \mathbf{e_2}\}$ be a local positively-oriented orthonormal basis for TM, then

$$\begin{aligned} B = B \cdot \mathrm{vol}_M(\mathbf{e}_1, \mathbf{e_2}) &= \iota^*\mathcal{F}(\mathbf{e}_1, \mathbf{e}_2)) \\ &= \mathcal{F}(\mathbf{e}_1, \mathbf{e}_2) \\ &= *\mathcal{F}(\mathbf{n}) \\ &= g((*\mathcal{F})^\sharp, \mathbf{n}) \\ &= g(\mathbf{B}, \mathbf{n}) = B^\perp. \end{aligned}$$

By an abuse of notation, we simply write

$$\mathcal{F} \text{ “} = \text{” } \iota^*\mathcal{F}$$

for the magnetic 2-form felt by a charged particle confined to M.

3.2 Landau operator on surfaces

The *quantum* mechanical description of electron motion on M subject to a magnetic field $\mathcal{F}$ requires the following U(1)-gauge-theoretic ingredients.

There is a smooth Hermitian line bundle, $\mathcal{L}^\nabla \to M$, whose connection ∇ has curvature 2-form $\mathcal{F} = B \cdot \mathrm{vol}_M$. The *Landau operator* associated to ∇ is the connection Laplacian

$$H_B = \nabla^*\nabla.$$

The ambiguity in the choice of ∇ (with curvature $B \cdot \mathrm{vol}_M$) is discussed later.

For simplicity, assume M is contractible. So $\mathcal{L}^\nabla$ can be trivialized (global gauge choice), thereby identifying $\mathcal{L}^\nabla \cong M \times \mathbb{C}$, and the L^2-sections with the Hilbert space of functions

$$L^2(M; \mathcal{L}^\nabla) \cong L^2(M) \equiv L^2(M; \mu_{\mathrm{vol}_M}) \tag{7}$$

Similarly, smooth sections $\cong C^\infty(M)$.

Once a gauge choice is made, ∇ is exhibited as the operator

$$\nabla \cong d - i\mathcal{A} : C^\infty(M) \to \Omega^1(M),$$

and $\mathcal{A}$ is called the *connection 1-form*. Switching to a different gauge changes the identification (7), implemented by a unitary map $L^2(M) \to L^2(M)$ of multiplication by some smooth function $U : M \to \mathrm{U}(1)$. By contractibility of M,

$$U = e^{i\Lambda}, \qquad \Lambda : M \to \mathbb{R}.$$

The connection 1-form with respect to the new gauge is

$$\tilde{\mathcal{A}} = \mathcal{A} + d\Lambda.$$

Let $\nabla, \tilde{\nabla}$ be any two connections with the same curvature form $\mathcal{F}$. With respect to any gauge choice, their respective connection 1-forms satisfy

$$d\tilde{\mathcal{A}} - d\mathcal{A} = 0.$$

By the Poincaré Lemma and contractibility of M, there is a 0-form Λ such that $\tilde{\mathcal{A}} = \mathcal{A} + d\Lambda$, and therefore a unitary gauge transformation $U = e^{i\Lambda}$ relates the two connections.

Thus, the magnetic field $B\cdot\mathrm{vol}_M$ is sufficient[3] for specifying the Landau operator $H_B = \nabla^*\nabla$, *up to gauge equivalence*. Physically measurable quantities must be gauge-*independent* quantities associated to H_B.

3.3 Simplified geometry, clean model

Let M = Euclidean/complex plane, so $\mathbb{R}^2$ acts on it by translation. The ordinary Laplacian $-\nabla^2$ on functions is translation invariant (under pullback). Its spectrum is easily found to be $[0,\infty)$ by Fourier transform.

Suppose $B(z) = b \in \mathbb{R}$ is constant (i.e. uniform $\perp$ field). Then $\mathcal{F} = b\cdot\mathrm{vol}_M$ is also translation invariant. Does this mean that H_B is translation invariant? How do translations act on sections of $\mathcal{L}^\nabla$? Before addressing these questions, let us compute the spectrum of H_B as follows.

Pick an origin $\mathcal{O}$. In Cartesian/complex coordinates, the (rotationally) "*symmetric gauge*" choice has[4]

$$\mathcal{A} = \frac{b}{2}(xdy - ydx) = \frac{ib}{4}(zd\bar{z} - \bar{z}dz), \qquad z = x + iy,$$

Evidently,

$$d\mathcal{A} = b\cdot dx\wedge dy = b\cdot\mathrm{vol}_M$$

holds. It is instructive to consider the Dirac operator on M coupled to $\mathcal{L}^\nabla$,

$$D_b = -2i\begin{pmatrix} 0 & \partial - \frac{b}{4}\bar{z} \\ \bar{\partial} + \frac{b}{4}z & 0 \end{pmatrix}$$

Then

$$D_b^2 = -4\begin{pmatrix} 0 & \partial - \frac{b}{4}\bar{z} \\ \bar{\partial} + \frac{b}{4}z & 0 \end{pmatrix}^2 = \begin{pmatrix} H_b - b & 0 \\ 0 & H_b + b \end{pmatrix}. \tag{8}$$

Geometrically, D_b is the Dirac operator of M coupled to $\mathcal{L}^\nabla$. The above equation is actually the Lichnerowicz identity expressed in symmetric gauge and Cartesian coordinates.

It follows quite easily that:

- $\mathrm{Spec}(H_b) = (2\mathbb{N}+1)|b|$, called (*Landau levels*).
- Lowest LL is the Dirac kernel. In symmetric gauge:

$$\ker\left(\bar{\partial} + \frac{b}{4}z\right) = \overline{\mathrm{span}\left\{z^n e^{-\frac{b}{4}|z|^2} \;:\; n\in\mathbb{N}\right\}}, \qquad b > 0,$$
$$\ker\left(\partial - \frac{b}{4}\bar{z}\right) = \overline{\mathrm{span}\left\{\bar{z}^n e^{-\frac{|b|}{4}|z|^2} \;:\; n\in\mathbb{N}\right\}}, \qquad b < 0.$$

  This is the (anti-)*Bargmann–Fock* space: (anti-)holomorphic[5] part of a Gaussian-weighted-L^2-space.

[3] If, for example, M is not simply-connected, then holonomies (Aharonov–Bohm fluxes) also come into play.

[4] In general coordinates, there would be a factor of $|g| = \sqrt{\det(g_{ij})}$, which is 1 in Cartesian coordinates. This is helpful for keeping track of units of length.

[5] If we are studying electron motion in a magnetic field, we would need to account for its negative charge. The connection 1-form, thus curvature, would have an extra minus sign. Consequently, the LLL projection for $b > 0$ should be the anti-Bargman–Fock space.

- Let $P \equiv P(b)$ be the projection onto LLL eigenspace. Remarkably,

$$2\pi i \cdot \mathrm{Tr}[P_X, P_Y] = \begin{cases} +1, & b > 0, \\ -1, & b < 0. \end{cases} \tag{9}$$

  (Compare Bergman space example.) This has a coarse index-theoretic reason, which is important for its robustness. We provide an "elementary" calculation in Section 7, which seems to be unavailable in the literature.

3.4 Physics and geometric stability

Units.

- $H_B = \frac{\hbar^2}{2m_e}\nabla^*\nabla$.
- $\nabla = d - i\frac{q}{\hbar}\mathcal{A}$ where $q = -e$ is electron charge.
- So $\mathcal{A}, \mathcal{F}, \Lambda$ have $\frac{\hbar}{e}$ units,

$$U = e^{i\frac{e}{\hbar}\Lambda} : M \to \mathrm{U}(1) \qquad \text{(dimensionless).}$$

- Note that in $\mathcal{F} = B \cdot \mathrm{vol}_M$, the function B has Tesla dimension and vol_M has area dimension[6]. Altogether, $\mathcal{F}$ has the same dimensions as $\frac{\hbar}{e}$, and it is "infinitesimal magnetic flux through surface".
- LLL has energy

$$E_0 = \frac{\hbar^2}{2m} \cdot \frac{e}{\hbar}|b| = \hbar \underbrace{\frac{e|b|}{2m}}_{\omega_b},$$

  with ω_b the "harmonic oscillator frequency".
- Gaussian weight for LLL is

$$\exp(-\tfrac{e}{4\hbar}|b||z|^2),$$

  so magnetic length scale $\sim |b|^{-1/2}$. Roughly $26\mathrm{nm}/\sqrt{\#\mathrm{Tesla}}$.
- Later we will see that each Landau level has room for

$$\frac{e}{h}|b| \ \text{ electrons per unit area.}$$

- Material M has a certain electron density μ, so *filling fraction*

$$\nu \equiv \nu(b) = \frac{\mu h}{e}|b|^{-1}, \tag{10}$$

  is a *pure number which is controlled by varying* b.

[6]In coordinates, $\mathrm{vol}_M = g \cdot dx \wedge dy$, where $g = \sqrt{\det(g_{ij})}$ has units of area.

Let X (resp. Y) be right (resp. upper) half-plane. Adiabatically apply electric voltage between X^c and X, and measure ("classically") the induced current from Y^c to Y. The ratio is the *Hall conductance*

$$\sigma_{\text{Hall}} \equiv \sigma_{\text{Hall}}(b) \equiv \sigma_{\text{Hall}}(\nu),$$

measured as a function of the control parameter b. It should be understood as the off-diagonal part of the conductance matrix (tensor)

$$\sigma = \begin{pmatrix} \sigma_{xx} & -\sigma_{\text{Hall}} \\ \sigma_{\text{Hall}} & \sigma_{yy} \end{pmatrix},$$

with inverse matrix

$$\rho = \begin{pmatrix} \rho_{xx} & \rho_{\text{Hall}} \\ -\rho_{\text{Hall}} & \rho_{yy} \end{pmatrix} = \sigma^{-1}.$$

Experiment at low temperature and large $|b|$. *For all $\nu \approx$ small integer*,

$$\rho_{xx}(\nu) = 0 = \rho_{yy}(\nu), \qquad \rho_{\text{Hall}}(\nu) = \frac{h}{e^2}\frac{1}{\nu_{\text{int}}}, \qquad \forall \nu \approx \text{small integer}.$$

where ν_{int} is the integer part of ν. This implies that

$$\sigma_{\text{Hall}}(\nu) = \frac{e^2}{h}\nu_{\text{int}}, \qquad \forall\, \nu \approx \text{integer}. \tag{11}$$

That is, the measured result gets "rounded off to the nearest integer".

- The rounding off is of astounding accuracy! ($\sim 10^{-10}$).
- The sample can be prepared "quick and dirty". There are non-uniform B, non-flat M, holes, complicated (random?) electrostatic potentials V,...). Yet, σ_{Hall} remains integer-quantized, as though all these "small-scale complications" are invisible!

Classical model fails miserably. In a classical EM treatment (no $\hbar$ allowed!), one gets (see physics textbooks, Drude model)

$$\rho_{\text{Hall,classical}}(b) = \frac{1}{\mu e}b, \qquad \forall b \in \mathbb{R}$$
$$(= \tfrac{h}{e^2}\tfrac{1}{\nu}, \quad \forall \nu \in \mathbb{R}\setminus\{0\}).$$

This is almost always wrong, compared to experiment (Eq. (3.4)). In fact, it is only correct for for exactly integer ν, which is a *quantum* condition!

Quantum model succeeds.

- Given a general Hamiltonian H, let P be the spectral projection onto filled electron states. By QM linear response *theory*, in the P-state (i.e. the many-electron state at 0 temperature), the Hall conductance should be

$$\sigma_{\text{Hall}}(P) = -2\pi i \cdot \operatorname{Tr}\big(P[[X,P],[Y,P]]\big), \qquad (\times \frac{e^2}{h}), \tag{12}$$

see, e.g., [Elgart–Schlein, CPAM '04]. Also see Remark 6.1.

This formula is precisely the Toeplitz-commutator which we had been studying,

$$\begin{aligned} P\big[[X,P],[Y,P]\big] &= P[X,P][Y,P] - P[Y,P][X,P] \\ &= P(XP-PX)(YP-PY) - P(YP-PY)(XP-PX) \\ &= PXPYP - PYPXP \\ &= [P_X, P_Y]. \end{aligned} \tag{13}$$

- Note that (13) is skew-adjoint, thus it has purely imaginary trace. So σ_{Hall} in (12) is always a real number.

  If H is *real* (i.e., commutes with a complex conjugation T, usually thought of as time-reversal), then so is P, as well as $[P_X, P_Y]$. Then $\sigma_{\text{Hall}}(P)$ must vanish. So H must break time-reversal symmetry, in order for $\sigma_{\text{Hall}}(P)$ to have a chance of being nonzero. The coupling to a gauge field (which cannot be gauged away) precisely achieves this symmetry breaking.
- If we tweak the magnetic field in clean QH sample so that the filling factor is exactly $\nu = 1$, then P=Landau level projection (for charge $-e$ electrons), and it may be explicitly calculated that

  $$\sigma_{\text{Hall}}(P) = -2\pi i \cdot \text{Tr}[P_X, P_Y] = \pm 1. \qquad (\times \frac{e^2}{h})$$

  All methods I am aware of to do this calculation seem to require homogeneity and index theory (e.g. Avron–Seiler–Simon, Bellissard). I will explain a T-duality/families index calculation later (Eq. (25)), as well as an elementary one in Section 7.
- Each filled Landau level contributes the same Hall conductance. So the clean quantum model again correctly predicts the experiment, Eq. (11), when $\nu \in \mathbb{N}$.
- However, in clean QM model, it is hard to even make precise sense of non-integer $\nu \sim b^{-1}$, since the Landau levels are infinitely-degenerate eigenvalues.
- In a true "dirty sample", the Landau levels will broaden into spectral regions around the Landau levels. The edges of these regions are thought to comprise "localized states". Now it makes sense to vary $b^{-1} \sim \nu$ to approximate integer values, thereby "partially filling the Landau bands".
- **Plateaux**. Provided spectral gap survives at "filling factor ν", the $\sigma_{\text{Hall}}(P(\nu))$ formula holds, and $P = f(H_B + V)$ for smooth f, so that we are within regime of our Theorem:

  $$2\pi i \cdot \text{Tr}[P_X, P_Y] \in \mathbb{Z}.$$

  Thus, QM model correctly predicts rounding off from ν to ν_{int}.
- Putting back the units, we see that the plateaux in the experimentally measured Hall conductance gives us *macroscopic*, easy, reliable, robust access to the fundamental physical constant $\frac{e^2}{h}$.
- Large b is important, so that the finite sized sample is well modelled by infinite sized model above.
- **Caveat**. Near to the Landau levels, true spectral gaps are not expected to survive. Instead, one has "mobility gaps", and it is a technical endeavour to extend the above discussion to this setting (achieved in the literature in some special models).

- **Transitions**. Indeed, the entire argument *needs* to fall apart as ν traverses the "core" of the spectral region surrounding a Landau level. (Roughly, ν is close to "half-integer".) The "core" comprises "delocalized states", which are the ones contributing to the Hall conductance. *Transitions* between integral Hall conductances occur here, as different amounts of delocalized states become occupied.

Remark 3.2. If P has finite-rank, then so does $P_X P_Y$ and $P_Y P_X$, and cyclicity of the trace applies to make $\sigma_{\mathrm{Hall}}(P) = 0$.

If X, Y are bounded subsets, then $P_X P_Y$ and $P_Y P_X$ will be trace class by the methods of Section 6. For example, if M is a finite sized sample, this forces $\sigma_{\mathrm{Hall}}(P) = 0$. (In this case H has compact resolvent, so P is finite-rank.) The crucial point is that the expression $\sigma_{\mathrm{Hall}}(\cdot)$ does not correctly give the *bulk* Hall conductance of a finite sample M, not even approximately. It includes the contribution of spectrum localized near to the boundary ∂M ("edge states"). We should instead do a calculation $\sigma_{\mathrm{Hall}}(KPK)$ truncated to a certain region $K \subset M$ away from ∂M. Now, KPK is no longer a projection, and the identification of $\sigma_{\mathrm{Hall}}(\cdot)$ as a commutator in (13) no longer applies. Thus the bulk Hall conductance need not vanish, and is *no longer exactly integral*. Instead, when K and M are sufficiently large, the bulk Hall conductance will approximate the exactly-quantized $\sigma_{\mathrm{Hall}}(P_{\mathrm{infinite}})$.

This is precisely what is seen in experiments: large fields $|b| \gg 1$ are needed to make the finite-sized sample "macroscopic" with respect to the length scale $|b|^{-1/2}$, and the plateaux of quantized Hall conductance is seen in such regimes.

4 Index-theoretic perspective

- In [K+L+T, CMP '21–'22], we explained geometric nature of Landau levels: they are twisted Dirac operator kernels. This led to general bulk-boundary correspondence proof by coarse index methods.
- On closed M, Diracs have *Fredholm* index (# zero-energy modes) computed by Atiyah–Singer's topological formula.
- The slogan "analytic index=topological index" might suggest that geometry is unimportant for "low energy physics". But this is not true on non-closed manifolds.
- (Roe, '90s) Index theory exists even on *topologically trivial* noncompact spaces. *Large-scale geometry* controls index, small-scale geometry is ignored. For geometric operators, "low energy" (index) $\leftrightarrow$ "large-scale".

4.1 Index pairing

The LHS of (9) is a gauge-independent quantity. Actually,

$$[P] = \pm[\text{Dirac}] \in K_0(C^*_{\text{Roe}}(M)),$$

and (9) is index pairing of $[D]$ with coarse cohomology class associated to X, Y. Roe's index theorem [4.42 in *Coarse cohomology and index theory on complete Riemannian manifolds*] calculates this to be ± 1.

Technically, Roe studied considered ordinary Dirac operator D. It has no spectral gap around 0, so $[D]$ had to be defined abstractly in the *algebraic* K-theory of

$$\mathcal{B}_{\text{fin}}(M) \subset C^*_{\text{Roe}}(M) \subset \mathcal{B}(L^2(M; \mathcal{S})).$$

Ordinary D has $\mathbb{R}^2$-invariance, allowing Fourier transform calculation. This is done at K-theory level, with $[D]$ represented by *virtual* projections in *unitization*, with no obvious relation to spectral projections (what we care about). Notion of homotopy invariance is unclear.

For *twisted* Dirac, $[P_{\text{LLL}}]$ directly lies in topological subalgebra

$$\mathcal{B}(M) \subset C^*_{\text{Roe}}(M),$$

without need for auxiliary virtual projections/unitization. However, standard translation invariance is lost — we need something more subtle to do calculation.

5 Magnetic translation symmetry

Each $\gamma \in \text{Isom}(M)$ acts isometrically (on the right) on the manifold M. The curvature $d\mathcal{A} = b \cdot \text{vol}_M$ of $\mathcal{L}^\nabla$ is likewise invariant under pullback by $\text{Isom}^+(M)$, thus

$$d(\mathcal{A} - \gamma^*\mathcal{A}) = 0, \qquad \forall \gamma \in \text{Isom}^+(M).$$

Poincaré Lemma says that there exists a 0-form ϕ_γ, such that

$$d\phi_\gamma = \mathcal{A} - \gamma^*\mathcal{A},$$

For example, fix an origin z_0 and take

$$\phi_\gamma(z) = \int_{z_0}^{z} (\mathcal{A} - \gamma^* \mathcal{A}). \tag{14}$$

This is well-defined (path-independent) since the integrand is closed. *But it depends on a choice of gauge.*

The *magnetic isometries* are operators on $L^2(M)$, preliminarily defined as

$$U_\gamma := e^{i\phi_\gamma}\gamma^*, \qquad \gamma \in \mathrm{Isom}(M)^+. \tag{15}$$

The assignment

$$\mathrm{Isom}^+(M) \to \mathcal{U}(L^2(M)), \qquad \gamma \mapsto U_\gamma$$

is not a group homomorphism. Instead, the U_γ satisfy

$$U_{\gamma_1} U_{\gamma_2} = \sigma(\gamma_1, \gamma_2) U_{\gamma_1\gamma_2},$$

where

$$\begin{aligned} \sigma(\gamma_1,\gamma_2) &= \exp\big(i\int_{z_0}^{z}(\mathcal{A}-\gamma_1^*\mathcal{A}) + i\int_{z_0}^{z\cdot\gamma_1}(\mathcal{A}-\gamma_2^*\mathcal{A}) \\ &\qquad - i\int_{z_0}^{z}(\mathcal{A}-(\gamma_1\gamma_2)^*\mathcal{A})\big) \\ &= \exp\big(i\int_{z_0}^{z_0\cdot\gamma_1}(\mathcal{A}-\gamma_2^*\mathcal{A})\big) \\ &= e^{i\phi_{\gamma_2}(z_0\cdot\gamma_1)} \qquad\qquad \in \mathrm{U}(1). \end{aligned} \tag{16}$$

Thus, there is actually a *central extension*

$$1 \to \mathrm{U}(1) \to \widetilde{\mathrm{Isom}^+(M)} \xrightarrow{\pi} \mathrm{Isom}^+(M) \to 1, \tag{17}$$

and the magnetic isometry assignment, Eq. (15), is a section. The map

$$\sigma(\cdot,\cdot) : \mathrm{Isom}^+(M) \times \mathrm{Isom}^+(M) \to \mathrm{U}(1)$$

is the *group* 2-cocycle for this section. The U(1) factor just acts as scalar multiplication — it is the fibrewise gauge group action.

We may check that

$$(d - i\mathcal{A}) \circ U_\gamma = U_\gamma (d - i\mathcal{A}). \tag{18}$$

Also, $U_{-\gamma}$ is the adjoint of U_γ up to a phase factor. So $(d-i\mathcal{A})^*$ also commutes with U_γ. Thus $H_b = (d-i\mathcal{A})^*(d-i\mathcal{A})$, and also P, commutes with U_γ. Similarly for D_b.

To summarize: there is an action of $\widetilde{\mathrm{Isom}^+(M)}$ on $\mathcal{L}^\nabla$ (tensored with trivial spinor bundle), commuting with H_b (or D_b), and lifting the action of $\mathrm{Isom}^+(M)$ on the base manifold M.

Gauge-dependence. For a different gauge, with $\tilde{\mathcal{A}} = A + d\Lambda$ the connection 1-form, the magnetic isometries are instead

$$\tilde{U}_\gamma = e^{i\tilde{\phi}_\gamma}\gamma^*$$

where

$$\tilde{\phi}_\gamma = \phi_\gamma + \Lambda(z) - \Lambda(z_0) - \Lambda(z\cdot\gamma) + \Lambda(z_0\cdot\gamma).$$

It follows that

$$\tilde{U}_\gamma = \underbrace{e^{i(\Lambda(z_0\cdot\gamma)-\Lambda(z_0))}}_{\in \mathrm{U}(1)} \cdot \underbrace{e^{i\Lambda}e^{i\phi_\gamma}\gamma^* e^{-i\Lambda}}_{UT_\gamma U^{-1}}, \tag{19}$$

where $U = e^{i\Lambda}$ is the unitary gauge transformation.

We learn that:

- The formula, Eq. 15, for the magnetic isometries, respects gauge transformations only up to a γ-dependent phase. Consequently, for a different choice of gauge, the 2-cocycle σ will be modified by a coboundary.
- The choice of origin z_0 in (15) introduces another phase ambiguity in the U_γ operators.

We might think that an appropriate choice of gauge will change allow for $\sigma \equiv 1$, so that the magnetic isometry group is unitarily equivalent to a direct product

$$\widetilde{\mathrm{Isom}^+}(M) \overset{?}{\cong} \mathrm{U}(1) \times \mathrm{Isom}^+(M).$$

However, the cohomology class of σ is an obstruction to achieving this simplification.

5.1 (Non)commutativity

Now let M be the Euclidean plane. So $\mathrm{Isom}^+ \cong \mathbb{R}^2 \rtimes \mathrm{SO}(2)$. One way to see that $\widetilde{\mathrm{Isom}^+}(M)$ is a non-trivial central extension of $\mathrm{Isom}^+(M)$ is to restrict attention to the translation subgroup $\mathbb{R}^2 \subset \mathrm{Isom}^+(M)$. There is a sub-central-extension,

$$1 \to \widetilde{\mathbb{R}^2} \to \mathbb{R}^2 \to 1,$$

and the lifts $U_\gamma, \gamma \in \mathbb{R}^2$, are usually called *magnetic translations* (with respect to a gauge choice).

The failure of the $U_\gamma, \gamma \in \mathbb{R}^2$ to commute is

$$\begin{aligned}
U_{\gamma_1}U_{\gamma_2}U_{\gamma_1}^{-1}U_{\gamma_2}^{-1} &= \sigma(\gamma_1,\gamma_2)U_{\gamma_1\gamma_2}U_{\gamma_2\gamma_1}^{-1}\sigma(\gamma_2,\gamma_1)^{-1} \\
&= \exp\big(i(\phi_{\gamma_2}(z_0\cdot\gamma_1) - \phi_{\gamma_1}(z_0\cdot\gamma_2))\big) \\
&= \exp\Big(i\Big(\int_{z_0}^{z_0\cdot\gamma_1} - \int_{z_0\cdot\gamma_2}^{z_0\cdot\gamma_1\cdot\gamma_2} - \int_{z_0}^{z_0\cdot\gamma_2} + \int_{z_0\cdot\gamma_1}^{z_0\cdot\gamma_2\cdot\gamma_1}\Big)\mathcal{A}\Big) \\
&= \exp\Big(i\oint_{z_0\to z_0\cdot\gamma_1\to z_0\cdot(\gamma_1\gamma_2)\to z_0\cdot\gamma_2\to z_0}\mathcal{A}\Big) \\
&= \exp\big(ib\gamma_1\cdot\gamma_2)\big). \qquad (\text{Stokes}' \text{ theorem})
\end{aligned} \tag{20}$$

Subtle point: In the next-to-last line, the paths $z_0 \to z_0\cdot\gamma_1$ and $z_0\cdot(\gamma_1\gamma_2) \to z_0\cdot\gamma_2$ must be related by translation-by-γ_2 and path-reversal. Similarly for the other pair of sub-paths. Then the last equality involving the enclosed area is a Euclidean space property.

Thus, as long as $b \neq 0$, the magnetic translations cannot be made to commute. This calculation does not depend on phase ambiguities in the U_γ, and is, in particular, gauge-independent.

5.1.1 Abelian lattice of magnetic translations

Fix generators γ_1, γ_2 of a square lattice $\Gamma_b \subset \mathbb{R}^2$, of side length $\sqrt{\frac{2\pi}{|b|}}$. There is a restricted central extension,

$$1 \to \mathrm{U}(1) \to \widetilde{\Gamma_b} \to \Gamma_b \to 1.$$

Eq. (20) shows that the magnetic translations $U_\gamma, \gamma \in \Gamma_b$ commute with each other. However, this does not mean that

$$\Gamma_b \to \widetilde{\Gamma_b}, \qquad \gamma \mapsto U_\gamma,$$

is a homomorphism. By abstract arguments, it is true, though, that we can attach appropriate phases to the magnetic translations $U_\gamma, \gamma \in \Gamma_b$ (in any gauge) to obtain a homomorphic lift, thus

$$\widetilde{\Gamma_b} \cong \mathrm{U}(1) \times \Gamma_b. \tag{21}$$

(See Landau gauge discussion later for an explicit discussion.)

Note that the choice of isomorphism with a direct product (i.e. commuting lifts U_γ), is not unique. Nor is there a canonical choice. Specifically, we can always modify

$$U_\gamma \rightsquigarrow \alpha(\gamma) U_\gamma,$$

where $\alpha(\gamma)$ are phase assignments satisfying

$$\alpha(\gamma + \gamma') = \alpha(\gamma)\alpha(\gamma'), \qquad \forall\, \gamma, \gamma' \in \Gamma_b, \tag{22}$$

i.e., α is a character of Γ_b. The character space of Γ_b can be identified with a 2-torus $\mathbb{T}^2$.

Remark 5.1. The above discussion can be described in terms of group cohomology. The central extension is trivial in $H^2(\Gamma_b, \mathrm{U}(1))$, and there is $H^1(\Gamma_b, \mathrm{U}(1)) \cong \mathrm{Hom}(\Gamma_b, \mathrm{U}(1))$ worth of trivializing lifts.

5.2 Moduli

5.2.1 Landau gauge

Fix an origin $\mathcal{O}$. Let (x, y) be Cartesian coordinates centred at $\mathcal{O}$, such that

$$\gamma_1 = (\sqrt{\tfrac{2\pi}{|b|}}, 0), \quad \gamma_2 = (0, \sqrt{\tfrac{2\pi}{|b|}}).$$

The so-called *Landau gauge* has

$$\mathcal{A} = b\, x dy,$$

and one checks from Eq. (16) that

$$\sigma(\cdot, \cdot) : \Gamma_b \to \Gamma_b \to \mathrm{U}(1) \equiv 1.$$

This exhibits a splitting, Eq. (21). In any other gauge, the magnetic translations are unitarily equivalent to the Landau gauge one up to phase redefinitions. So the former can always be phase-redefined to obtain $\gamma \mapsto U_\gamma$ as a homomorphism.

As mentioned before, we can modify the (Landau gauge) magnetic translations by a character, without affecting their trivial cocycle σ. The explicit "source" of these freedoms is as follows:

- Eq. (14) says that in Landau gauge, changing z_0 to $z_0 + (\frac{k_x}{b}, \frac{k_y}{b})$ causes U_{γ_1} to be adjusted by a phase factor $\exp(ik_y\sqrt{\frac{2\pi}{|b|}})$, while U_{γ_2} is unaffected.
- If the origin for Landau gauge is changed to $\mathcal{O} + (\frac{k_x}{b}, \frac{k_y}{b})$, the "new Landau gauge" would have $\tilde{\mathcal{A}} = b(x - \frac{k_x}{b})dy$, and is therefore obtained from the "previous Landau gauge" by the gauge transformation $U(x,y) = e^{-ik_x y}$. According to Eq. (19), the magnetic translation U_{γ_2} would acquire an extra phase factor of $\exp(ik_x\sqrt{\frac{2\pi}{|b|}})$, while U_{γ_1} is unaffected.

Write $k = (k_y, k_x)$. The above says that a k_x/b horizontal shift of $\mathcal{O}$ and a k_y/b vertical shift of z_0 result in phase factors

$$U_\gamma \rightsquigarrow e^{i\langle k,\gamma\rangle} U_\gamma, \qquad \gamma \in \Gamma_b,$$

for the "Landau gauge magnetic translations".

These ambiguities are parametrized by a torus

$$k = (k_y, k_x) \in \left[0, \sqrt{2\pi|b|}\right]^2 /_\sim = \mathbb{T}^2,$$

which is precisely the character space for Γ_b, mentioned in (22). Note that the natural orientation on $\mathbb{T}^2$ is $dk_x \wedge dk_y$ in these coordinates.

Remark 5.2. The upshot is that the total space of the bundle $\mathcal{L}^\nabla$ admits a genuine Γ_b action (by magnetic translations), lifting the action of Γ_b on the base M, and commuting with the Landau operator. However, there is no canonical choice for this action. Instead, there is a $\mathbb{T}^2$-torsor ("moduli") of possible choices.

5.3 T-duality

Let $T^2 = M/\Gamma$ be a fundamental domain, which is an affine 2-torus. Regard T^2 as a square in M of side length $\sqrt{\frac{2\pi}{|b|}}$, with opposite edges identified. Do not confuse T^2 with $\mathbb{T}^2$ — they are dual to each other.

Each choice of Γ_b-action (a point in the moduli). The quotient

$$\mathcal{L}^\nabla/\Gamma_b$$

defines a line bundle with connection,

$$\mathcal{L}^{\nabla^{(k)}} \to T^2.$$

Concretely, the phase functions $e^{i\phi_{\gamma_1}}, e^{i\phi_{\gamma_2}}$ are the U(1)-valued transition functions needed for patching fibres on opposite sides of the square together.

Over T^2, the line bundle curvature integrates to $\pm 2\pi$. The Dirac quantization condition is satisfied, and topologically, the line bundle $\mathcal{L} \to T^2$ has Chern class being a generator of $H^2(T^2;\mathbb{Z})$. This is independent of the choice k of Γ_b-action.

However, the *global holonomies* of $\nabla^{(k)}$ are different. Recall that shifting the choice of action by the parameter $k = (k_y, k_x) \in \mathbb{T}^2$ changes the fibre identifications when gluing edges of the square to form cycles of T^2. Geometrically, a shift by k introduces an extra flat connection with holonomies $(e^{i\sqrt{\frac{2\pi}{|b|}}k_y}, e^{i\sqrt{\frac{2\pi}{|b|}}k_x})$ around the x-cycle and the y-cycle, respectively

Overall, there is a family of quotient line bundles,

$$\mathcal{L}^{\nabla^{(k)}} \to T^2, \qquad k \in \mathbb{T}^2,$$

and a corresponding *family* of Landau operators

$$H_{b,k} = (\nabla^{(k)})^* \nabla^{(k)}, \qquad k \in \mathbb{T}^2.$$

Similarly, we have the family of twisted Dirac operators,

$$D_{b,k}, \qquad k \in \mathbb{T}^2,$$

acting on the trivial spinor bundle of T^2 tensored with $\mathcal{L}^{\nabla^{(k)}}$.

5.3.1 Fourier transform of Landau levels

Lichnerowicz identity (8) still holds,

$$D_{b,k}^2 = \begin{pmatrix} H_{b,k} - b & 0 \\ 0 & H_{b,k} + b \end{pmatrix},$$

as k does not introduce any curvature corrections.

Each $H_{b,k}$ and $D_{b,k}$ is elliptic over the closed manifold T^2, with discrete spectrum. So the lowest Landau level of $H_{b,k}$ is a finite-dimensional eigenspace, identified with the $\ker(D_{b,k})$. Varying $k \in \mathbb{T}^2$, we obtain a vector bundle $\{\ker(D_{b,k})\}_{k\in\mathbb{T}^2}$ of Dirac kernels, which we should think of as the "Fourier transform" of $\ker(D_b)$ with respect to the abelian lattice Γ_b.

Details. We write $\mathcal{H}_k$ for the Hilbert space of L^2-sections of $\mathcal{L}^{\nabla^{(k)}} \to T^2$. Each $\mathcal{H}_k$ is isomorphic to $L^2(T^2)$ by picking a local trivialization over the interior of the square. (We can do this concurrently for all k by using a global gauge over M.) So there is a Hilbert space bundle,

$$\mathcal{H} \to \mathbb{T}^2$$

with fibres $\mathcal{H}_k$. With a gauge choice, one has

$$L^2(\mathbb{T}^2; \mathcal{H}) \cong L^2(\mathbb{T}^2) \otimes L^2(T^2).$$

Let ψ be a Schwartz class section of $\mathcal{L}^\nabla \to M$. For each k, the "k-quasiperiodic part" of ψ is given by the *Bloch sum*

$$\psi_k := \sum_{\gamma\in\Gamma_b} e^{i\langle k,\gamma\rangle} U_\gamma \psi.$$

Since ψ_k is invariant under the k-th Γ_b-action, it determines an element of $\mathcal{H}_k$. This construction is then extended to general L^2-sections of $\mathcal{L}^\nabla \to M$, and defines a unitary map

$$\mathcal{U} : L^2(M; \mathcal{L}^\nabla) \to L^2(\mathbb{T}^2; \mathcal{H}).$$

For $b = 0$, this is known as the *Bloch–Floquet* transform (of functions, global gauge assumed), ubiquitous in solid-state physics.

Under $\mathcal{U}$, the Γ_b-invariant operator H_b is transformed into the *family* $\{H_{b,k}\}_{k\in\mathbb{T}^2}$, with each $H_{b,k}$ acting on the $\mathcal{H}_k$ fibre. Geometrically, there is a fibre bundle

$$\pi_2 : T^2 \times \mathbb{T}^2 \to \mathbb{T}^2,$$

and H is turned into a $\mathbb{T}^2$-parametrized family of elliptic operators on the fibre T^2. Likewise for the Dirac operator D_b. We are interested in what $\mathcal{U}$ does to the LLL/Dirac kernel.

The following can be computed explicitly in Landau gauge:

- $\ker(D_{b,k})$ is one-dimensional, for each $k \in \mathbb{T}^2$, and is $\pm$-graded depending on the sign of b. Thus the (virtual) kernel bundle, i.e., the *index bundle*,

  $$\mathrm{Ind}_{\mathbb{T}^2}(D_b) = \pm\{\ker(D_{b,k})\}_{k\in\mathbb{T}^2}$$

  is well-defined over $\mathbb{T}^2$, and represents the families index in $K^0(\mathbb{T}^2)$.
- The Chern class of $\{\ker(D_{b,k})\}_{k\in\mathbb{T}^2}$ is a generator of $H^2(\mathbb{T}^2;\mathbb{Z})$ (independent of b).

A more abstract calculation proceeds as follows. Start from the bundle $\mathcal{L}^{\nabla^{(0)}} \to T^2$, which has Chern character

$$\mathrm{Ch}(\mathcal{L}^{\nabla^{(0)}}) = 1 + \frac{b}{2\pi} dx \wedge dy,$$

Over $T^2 \times \mathbb{T}^2$, the so-called *Poincaré line bundle*

$$\mathcal{P} \to T^2 \times \mathbb{T}^2,$$

has the property that its restriction to each $T^2\times\{k\}$ is flat with holonomies $(e^{i\sqrt{2\pi/|b|}k_y}, e^{i\sqrt{2\pi/|b|}k_x})$. Think of the connection form as

$$k_y dx + k_x dy$$

with x, y the local Cartesian coordinates on T^2. So $\mathcal{P}$ is not flat overall, and has Chern character

$$\mathrm{Ch}(\mathcal{P}) = 1 - \frac{1}{2\pi}(dk_y \wedge dx + dk_x \wedge dy) + \frac{1}{4\pi^2}(dk_y \wedge dx \wedge dk_x \wedge dy).$$

Recall that each $H_{b,k}$ acts on sections of $\mathcal{L}^{\nabla^{(k)}} \to T^2$. Altogether, the collection of line bundles,

$$\{\mathcal{L}^{\nabla^{(k)}}\}_{k\in\mathbb{T}^2}$$

comprises the restrictions of the following bundle over $T^2 \times \mathbb{T}^2$,

$$\pi_1^*\mathcal{L}^{\nabla^{(0)}} \otimes \mathcal{P}.$$

For the $D_{b,k}$, we tensor with the trivial spinor bundle as usual.

Now the families index formula gives

$$
\begin{aligned}
&\pm \operatorname{Ch}(\{\ker(D_{b,k})\}_{k\in\mathbb{T}^2}) \\
&\quad = \operatorname{Ch}(\operatorname{Ind}_{\mathbb{T}^2}(D_b)) && (23)\\
&\quad = \int_{T^2} \operatorname{Ch}(\pi_1^*\mathcal{L}^{(0)} \otimes \mathcal{P}) \\
&\quad = \int_{T^2} (1 + \frac{b}{2\pi} dx \wedge dy) \wedge \Big(1 - \frac{1}{2\pi}(dk_y \wedge dx + dk_x \wedge dy) && (24)\\
&\qquad\qquad\qquad\qquad\qquad + \frac{1}{4\pi^2}(dk_y \wedge dx \wedge dk_x \wedge dy)\Big) \\
&\quad = \pm\big(1 \pm \frac{1}{2\pi|b|} dk_x \wedge dk_y\big) && (25)
\end{aligned}
$$

This gives the result that the Dirac kernel bundle/Fourier transformed LLL has Chern number ± 1.

Remark 5.3. For the connection on $\{\ker(D_{b,k})\}_{k\in\mathbb{T}^2}$, *and*, we might use the one induced in its role as a subbundle of the trivialized Hilbert bundle $\mathcal{H}$. The latter trivialization refers to a gauge choice. So the connection on $\{\ker(D_{b,k})\}_{k\in\mathbb{T}^2}$, and even its curvature, is not canonical. But the integrated curvature, i.e. Chern number, is independent of these choices.

Scale invariance. Suppose we had chosen a coarser sublattice $\Gamma_{b,R}$ of Γ_b, generated by $R\gamma_1$ and $R\gamma_2$, where $R \in \mathbb{N}_{>1}$. Then T^2 has area $2\pi R^2/|b|$, while $\mathbb{T}^2$ has area $2\pi|b|/R^2$. The calculation above would give

$$
\operatorname{Ch}(\{\ker(D_{b,k})\}_{k\in\mathbb{T}^2}) = R^2 \pm \frac{R^2}{2\pi|b|} dk_x \wedge dk_y,
$$

and thus the same ± 1 Chern number. The rank, however, jumps to R^2.

The invariance of the Chern number to the choice of lattice scale is of great importance, since the latter is, after all, just a mathematical choice. This indicates that the Chern number is a "large-scale" invariant. Indeed, we may consider sufficiently large R, and identify the expression of integrating the curvature over $\mathbb{T}^2$ with the expression $\operatorname{Tr}[P_X, P_Y]$.

Remark 5.4. If $b = 0$, we get an index bundle of Chern class 1 and virtual rank 0. Stabilization is needed. Neither graded part of the index bundle can be identified with a physical P.

5.4 Baum–Connes assembly map

Notice that $T^2 = B\Gamma_b$ (classifying space). The Baum–Connes assembly map μ_{Γ_b} gives an isomorphism

$$
K_0(T^2) \xrightarrow{\mu} K_0(C_r^*(\Gamma_b)) \overset{\text{Fourier}}{\cong} K^0(\mathbb{T}^2) \overset{\text{Ch}}{\cong} H^2(\mathbb{T}^2;\mathbb{Z}) \cong \mathbb{Z}^2.
$$

We evaluated this for the (K-homology class of the) Dirac operator twisted by $\mathcal{L}^{\nabla^{(0)}}$.

When lifted to the Dirac operator on $M = E\Gamma_b$ twisted by $\mathcal{L}^{\nabla}$, we get a Γ_b-equivariant elliptic operator with $C_r^*(\Gamma_b)$-index. We made this concrete with Fourier transform.

6 Finite-propagation, locally trace class operators

We wish to consider the QHE on more general M and B, so we need a more general non-equivariant index. Which operator algebra is suitable for this?

- A (self-adjoint) Dirac operator D on M propagates at unit speed. So unitary e^{iDt} propagates distance $|t|$. Schwartz function of D has
$$f(D) = \frac{1}{2\pi} \int \hat{f}(t) e^{itD}\, dt$$
with $\hat{f}$ Schwartz. Then $f(D)$ is approximated by finite-propagation operators (better than in operator norm).
- With a spectral gap, low energy spectral subspace has projection
$$P = f(D)$$
being approximately finite-propagation. Think of P as a big matrix (in "position basis") whose entries decrease rapidly with distance from diagonal.
- If K, K' compact, then KPK' is trace-class (restrict smooth integral kernel to compact subset).

Roe considered

$$\mathcal{B}_{\text{fin}}(M) = \{\text{locally trace class, finite propagation}\}$$
$$C^*_{\text{Roe}}(M) = \overline{\mathcal{B}_{\text{fin}}(M)}^{\text{op}}.$$

For even-dimensional M, he defined *coarse index* of D as an element of

$$K_0(C^*_{\text{Roe}}(M)), \qquad \text{or} \qquad K_0^{\text{alg}}(\mathcal{B}_{\text{fin}}(M)),$$

thus represented by (differences of) projections in the (unitized) algebras.

Recall that $D = \begin{pmatrix} 0 & D^*_{+-} \\ D_{+-} & 0 \end{pmatrix}$. If D has isolated interval $I \ni 0$ in spectrum, the spectral projection $P = f(D)$ for some even characteristic function f on I, so

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix} \in C^*_{\text{Roe}}(M).$$

Then

$$\text{Ind}(D) = [P_+] - [P_-] \in K_0(C^*_{\text{Roe}}(M)).$$

Turning on gauge field introduces lower-order term in D_b, without changing the K-theoretic index.

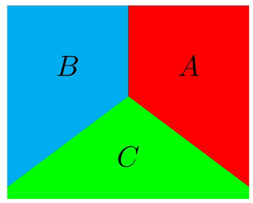
$$\text{Ind}(D_b) = [P_{\text{LLL}}] \in K_0(C^*_{\text{Roe}}(M))$$

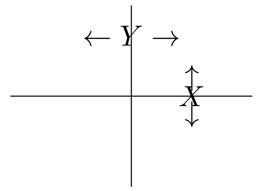
This can be done continuously in $b \neq 0$ (contrast Dirac quantization constraint for M_{cpt}).

But... what does a non-trivial coarse index "count"?

- In M_{cpt} case, P_+, P_- are trace-class kernel projections, thus finite-rank. So take supertrace,
$$\text{STr}(P_+ \ominus P_-) = \dim P_+ - \dim P_- = \text{Fredholm index}(D).$$
- For noncompact M, the $P_\pm$ are not trace class, so we need "higher traces" to "measure" them. The technology of *cyclic cohomology* and NCG studies this systematically; however, we cannot work on the whole $C^*_{\text{Roe}}(M)$.

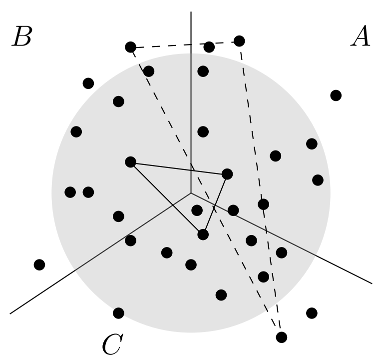
6.1 Pairing P with coarse partition





For simplicity, assume (incorrectly) that $P = f(D)$ is in $\mathcal{B}_{\text{fin}}(M)$. The following trace-class assumptions for the usage of our main theorem are satisfied:

- $P_X - P_X^2 = PXPX^cP$ is supported near $X \cap X^c$. Similarly, $P_Y - P_Y^2$ supported near $Y \cap Y^c$. Thus
$$(P_X - P_X^2)(P_Y - P_Y^2)$$
is supported near origin, thus trace class. Note that neither $P_X - P_X^2$ nor $P_Y - P_Y^2$ is Hilbert–Schmidt (why?). So the fact that their product is trace-class is somewhat remarkable.
- To show that $[P_X, P_Y]$ is trace class as well, we consider disjoint partition[7],
$$A = X, \qquad B = X^c \cap Y, \qquad C = X^c \cap Y^c.$$
Notice that $APBPCP$ is supported near triple intersection, so it is trace class, with trace given by summing loop amplitudes near intersection:



So the following "nett chiral loop amplitude" makes sense,
$$\mathrm{Tr}(APBPCP + \text{antisymm})$$
It will be convenient to write, for any three functions f_1, f_2, f_3,
$$[f_1, f_2, f_3]_P := \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) \cdot f_1 P f_2 P f_3 P.$$
Note that when one of the arguments is the identity function, then (say)
$$\begin{aligned} [f_1, f_2, 1]_P &= f_1Pf_2P + f_2Pf_1P + Pf_1Pf_2P - Pf_2Pf_1P - f_2Pf_1P - f_1Pf_2P \\ &= [P_{f_1}, P_{f_2}]. \end{aligned} \tag{26}$$

[7] An alternative approach uses Eq. (13), see Section 7.1.

We are particularly interested in

$$\begin{aligned}
\mathrm{Tr}(APBPCP + \text{antisymm}) &= [A,B,C]_P \\
&= [A,B,1-A-B]_P \\
&= [A,B,1]_P \qquad\qquad \text{(antisymmetry)} \\
&\overset{(26)}{=} [P_A,P_B]. \qquad (27)
\end{aligned}$$

By setting $A = X, B = X^c \cap Y$, it may be checked that

$$[P_X, P_Y] = 2[P_A, P_B] + \text{traceless}. \qquad (28)$$

(This uses a cobordism invariance argument, see Section 6.2 for a clue, and [arXiv:2308.02819] for details.)

In view of (27), we learn that $[P_X, P_Y]$ is indeed trace class.

Our theorem now applies to give

$$\sigma_{\mathrm{Hall}} = -2\pi i \cdot [P_X, P_Y] \in \mathbb{Z}.$$

At abstract level, a coarse partition $\{A, B, C\}$ determines a *coarse 2-cochain* over M, and then a cyclic 2-cocycle on $\mathcal{B}_{\mathrm{fin}}(M)$. This descends to a cyclic cohomology pairing with the algebraic K_0-theory of $\mathcal{B}_{\mathrm{fin}}(M)$ [Connes '85].

6.1.1 Extension of arguments to Fréchet algebra containing P

In reality, P is not finite-propagation. Certainly it lies in $C^*_{\mathrm{Roe}}(M)$, but higher-trace pairing becomes ill-defined. The problem is that operator norm is too flabby to control traces.

- We showed that P lies in a certain *Fréchet* subalgebra $\mathcal{B}(M) \subset C^*_{\mathrm{Roe}}(M)$ comprising *rapid decrease* operators in trace-norm sense,

  $$\sup_{\mathrm{diam}(K),\mathrm{diam}(K')\leq 1} ||KLK'||_{\mathrm{Tr}}(1 + d(K,K'))^{\nu} < \infty, \qquad \forall \nu.$$

  Rapid decrease controls loop amplitudes against volume growth.
- We prove that arguments of Section 6.1 work for $P \in \mathcal{B}(M)$.
- En route, we construct localized versions

  $$\mathcal{B}(M; Z), \qquad Z \subset M,$$

  and prove ideal property with respect to "large-scale excisiveness". Thus, there is a "locally trace-class" version of (localized) Roe C^*-algebras. Unlike the (localized) Roe C^*-algebras, *the construction is not based on coarsely invariant concepts, but a rather more restrictive large-scale geometry concept!*
- Specifically, assume that M has bounded geometry and *polynomial* volume growth (say). Then we need to use the notion of *polynomial* excisiveness for subsets,

  $$\bigcap_{n=0}^{q} B_r(Z_n) \subset \left(\bigcap_{n=0}^{q} Z_n\right)_{r^\mu}.$$

6.2 Cobordism invariance

Let $f_1 + f_2 + f_3 = 1$. Modify this "partition-of-unity" by a function g to

$$(f_1 - g) + (f_2 + g) + f_3 = 1.$$

The difference in their bracket with P is

$$\begin{aligned}
[f_1 - g, f_2 + g, f_3]_P - [f_1, f_2, f_3]_P &= [f_1, g, f_3]_P - [g, f_2, f_3]_P \\
&= [f_1 + f_2, g, f_3]_P \\
&= [1, g, f_3]_P \qquad\qquad ([g, f_3, f_3]_P = 0) \\
&\overset{(26)}{=} [P_g, P_{f_3}]. \qquad (29)
\end{aligned}$$

Suppose

- $\mathrm{supp}(f_1) \cap \mathrm{supp}(f_2) \cap \mathrm{supp}(f_3)$ is compact. ("Coarse partition-of-unity").
- $\mathrm{supp}(g) \cap \mathrm{supp}(f_3)$ is compact, so that

  $$f_1 - g, f_2 + g, f_3$$

  is another coarse partition-of-unity, "cobordant" to f_1, f_2, f_3 via g.

Then (29) is trace class, and implies the following "cobordism invariance"

$$\mathrm{Tr}[f_1, f_2, f_3]_P = \mathrm{Tr}[f_1 - g, f_2 + g, f_3]_P. \qquad (30)$$

For example, we can modify a *disjoint* partition A, B, C by "smoothening out" the interface between A, B (where the modification takes place near the interface). Iterate this for the B, C and the C, A interfaces. Eq. (30) implies that if we replace A, B, C with smooth partitioning functions $\tilde{A}, \tilde{B}, \tilde{C}$, we still have

$$\mathrm{Tr}[A, B, C]_P = \mathrm{Tr}[\tilde{A}, \tilde{B}, \tilde{C}]_P. \qquad (31)$$

Remark 6.1. Instead of taking X, Y to be χ_X, χ_Y, it is usual to use smoothened characteristic functions $\tilde{X}, \tilde{Y}$ in the Hall conductance formula (12). Equivalently, as in (13),

$$\sigma_{\mathrm{Hall}}(P) = -2\pi i \cdot \mathrm{Tr}[P_{\tilde{X}}, P_{\tilde{Y}}].$$

The expressions (28) and (27) still hold, with A, B, C replaced by the corresponding smoothened characteristic functions $\tilde{A}, \tilde{B}, \tilde{C}$ (summing to 1). The invariance property (31) allows us to conclude that

$$\mathrm{Tr}[P_{\tilde{X}}, P_{\tilde{Y}}] = \mathrm{Tr}[P_X, P_Y].$$

7 Explicit index calculation for Landau level

Subsequently we work with the standard (holomorphic) Bargmann space

$$\mathcal{H}_{\mathrm{Barg}} = \overline{\mathrm{span}\left\{ z^n e^{-\frac{|z|^2}{2}} \,:\, n \in \mathbb{N} \right\}},$$

which is the LLL eigenspace for $b = 2$. We write P for the orthogonal projection from $L^2(\mathbb{C}) = L^2(\mathbb{R}^2; \mu_{\mathrm{Leb}}) \to \mathcal{H}_{\mathrm{Barg}}$. It is known that the integral kernel of P, with respect to Lebesgue measure, is

$$p(z,w) = \frac{1}{\pi} e^{-\frac{|z|^2+|w|^2}{2}} e^{z\bar{w}}. \tag{32}$$

$$= \frac{1}{\pi} e^{-\frac{1}{2}|z-w|^2} e^{iw\wedge z}, \qquad w \wedge z := w_x z_y - z_x w_y, \tag{33}$$

Note that $p(\cdot,\cdot)$ is smooth and rapidly decaying away from the diagonal.

In [2308.02819], we proved that for general idempotents $P = P^2 \in \mathcal{B}(\mathbb{C}) \subset C^*_{\mathrm{Roe}}(\mathbb{C})$, the following pairing formula makes sense, and is integral,

$$2\pi i \cdot \mathrm{Tr}[P_X, P_Y] \in \mathbb{Z}. \tag{34}$$

Here, X, Y are polynomially transverse half-planes, and $P_X = P\chi_X P$ etc. are the compressions to the Bargmann space.

7.1 Explicit "elementary" calculation for Bargmann projection.

Our goal is to calculate Eq. (34) explicitly, for P the Bargmann space projection, and X, Y the right half-plane and upper half-plane respectively. In this case, the expression (34) is a pairing of the Dirac coarse index (represented by P) with a cyclic/coarse cohomology class associated to X, Y. Roe's abstract coarse index theorem can then be used to deduce

$$2\pi i \cdot \mathrm{Tr}[P_X, P_Y] = 1. \tag{35}$$

Let us compute this directly by "elementary" means.

Since we have the explicit expression for $p(\cdot,\cdot)$ from (32), we might try to directly integrate the kernel of $[P_X, P_Y]$ along the diagonal. But the terms $P_X P_Y$ and $P_Y P_X$ are generally not trace class, and their kernels will have divergent integrals along the diagonal.

To circumvent this, we use the alternative formula Eq. (13), which reads

$$[P_X, P_Y] = P\big[[X,P],[Y,P]\big] = P[X,P][Y,P] - P[Y,P][X,P]. \tag{36}$$

Since

$$[X,P] = XPX + XPX^c - XPX - X^cPX = XPX^c - X^cPX \tag{37}$$

decays rapidly away from $X \cap X^c$, while $[Y,P]$ similarly decays rapidly away from $Y \cap Y^c$, the products $[X,P][Y,P]$ and $[Y,P][X,P]$ decay rapidly away from the origin. Thus the latter are trace class, by the locally trace class and approximately finite-propagation properties of $P \in \mathcal{B}(\mathbb{C})$ (details provided in [arXiv:2308.02819]). We note in passing that this also shows, via Eq. (36), that $[P_X, P_Y]$ is trace class. Crucially, the extra P in (13) means that Lidskii's theorem does not apply, and it is possible for $[P_X, P_Y]$ to have non-vanishing trace.

In summary,

$$\begin{aligned}\mathrm{Tr}[P_X, P_Y] &= \mathrm{Tr}\big(P\big[[X,P],[Y,P]\big]\big) &(38)\\ &= \int_{u,v,w\in\mathbb{C}} p(u,v)p(v,w)p(w,u)\big(X(v)-X(w)\big)\big(Y(w)-Y(u)\big)\ -(X\leftrightarrow Y), &(39)\end{aligned}$$

where $p(\cdot,\cdot)$ given by (32). This integral now makes sense, but to compute it, we must invoke a symmetry argument, inspired by §4 of Avron–Seiler–Simon.

From the formula (33), we deduce that $p(\cdot,\cdot)$ has the translation covariance

$$p(z-t, w-t) = U_t(z)p(z,w)U_t(w)^{-1}, \qquad t\in\mathbb{R}^2\cong\mathbb{C}, \tag{40}$$

where $U_t:\mathbb{R}^2\to \mathrm{U}(1)$ is the gauge transformation[8]

$$U_t(z) = \exp\big(iz\wedge t\big).$$

This implies that

$$p(u,v)p(v,w)p(w,u) = p(u-w, v-w)p(v-w,0)p(0,u-w), \qquad u,v,w\in\mathbb{C}.$$

Then the formula (39) simplifies to

$$\begin{aligned}\mathrm{Tr}[P_X, P_Y] &= \int_{u,v,w\in\mathbb{C}} p(u,v)p(v,w)p(w,u)\big(X(v)-X(w)\big)\big(Y(w)-Y(u)\big)\ -(X\leftrightarrow Y)\\ &= \int_{u,v,w\in\mathbb{C}} p(u-w,v-w)p(v-w,0)p(0,u-w)\big(X(v)-X(w)\big)\big(Y(w)-Y(u)\big)\ -(X\leftrightarrow Y)\\ &\overset{\text{substitute}}{=} \int_{u,v,w\in\mathbb{C}} p(u,v)p(v,0)p(0,u)\big(X(v+w)-X(w)\big)\big(Y(w)-Y(u+w)\big)\ -(X\leftrightarrow Y)\end{aligned} \tag{41}$$

Using the rapid decrease of $p(\cdot,\cdot)$, we may do the w-integral first. This integral is independent of P, and has a simple geometric meaning:

$$\begin{aligned}&\int_{w\in\mathbb{C}} \big(X(v+w)-X(w)\big)\big(Y(w)-Y(u+w)\big)\ -(X\leftrightarrow Y)\\ &= \int_{w_x\in\mathbb{R}}\big(\chi_{[0,\infty)}(v_x+w_x)-\chi_{[0,\infty)}(w_x)\big)\int_{w_y\in\mathbb{R}}\big(\chi_{[0,\infty)}(w_y)-\chi_{[0,\infty)}(u_y+w_y)\big)\ -(x\leftrightarrow y)\\ &= (v_x)(-u_y)-(v_y)(-u_x)\\ &= u\wedge v.\end{aligned} \tag{42}$$

[8]The covariance property of p is just a restatement of the magnetic translation symmetry of P, and the gauge transformation $U_t(\cdot)$ is that appearing in the formula (15) for magnetic translations.

Using this geometric identity, Eq. (41) becomes

$$
\begin{aligned}
&\operatorname{Tr}[P_X, P_Y] \\
&= \int_{u,v\in\mathbb{C}} p(u,v)p(v,0)p(0,u)u \wedge v && (43) \\
&\overset{(32)}{=} \frac{1}{\pi^3}\int_{u,v\in\mathbb{C}} e^{-(|u|^2+|v|^2)} e^{u\bar{v}} u \wedge v \\
&= \frac{1}{\pi^3}\int_{r_1,r_2\in[0,\infty)} r_1 r_2\, e^{-(r_1^2+r_2^2)} \int_{\theta_1,\theta_2\in[0,2\pi]} e^{r_1 r_2 e^{i(\theta_1-\theta_2)}} r_1 r_2 \frac{1}{2i}\big(e^{i(\theta_2-\theta_1)} - e^{-i(\theta_2-\theta_1)}\big) \\
&= \frac{1}{\pi^3}\frac{1}{2i}\int_0^\infty \int_0^\infty dr_1\, dr_2\, (r_1 r_2)^2 e^{-(r_1^2+r_2^2)} \int_0^{2\pi} d\theta_1 \int_{S^1} e^{r_1 r_2 \bar{\alpha}} (\alpha - \bar{\alpha}) \frac{d\alpha}{i\alpha} && (\alpha = e^{i(\theta_2-\theta_1)}) \\
&= \frac{1}{\pi^3}\frac{1}{2i}(2\pi)\int_0^\infty \int_0^\infty dr_1\, dr_2\, (r_1 r_2)^2 e^{-(r_1^2+r_2^2)} \cdot (2\pi i)\big(\frac{r_1 r_2}{i}\big) && \text{(residue theorem)} \\
&= \frac{1}{\pi^3}\frac{1}{2i}(2\pi)(2\pi)\Bigg(\underbrace{\int_{r\in[0,\infty)} dr\, r^3 e^{-r^2}}_{1/2}\Bigg)^2 \\
&= \frac{1}{2\pi i}. && (44)
\end{aligned}
$$

Remark 7.1. Let us replace $X = \chi_X$ with a smooth *switch function* of x. This means the left limit is 0, right limit is 1, with interpolating region within a vertical strip of finite width. Similarly for Y. Everything in this section still holds. (That the geometric identity (42) still holds deserves special mention.)

Remark 7.2. The expression $-i[P_X, P_Y]$, with smooth *switch functions* X, Y, was called the *adiabatic curvature* in Definition 6.3 of Avron–Seiler–Simon. It was not shown that $[P_X, P_Y]$ is trace class, so the authors had to truncate using boxes K_R of side length $2R$ centred at the origin, and considered $\lim_{R\to\infty} \operatorname{Tr}\big(K_R[P_X, P_Y]K_R\big)$. This limit was called *Hall charge transport*. The manipulations up to Eq. (43) work generally for "homogeneous" P with magnetic translation symmetry, and it was shown that the expression equals $\frac{1}{2\pi i}$ times the Fredholm index of $P\frac{z}{|z|}P$ (thus integral), the latter index being interpreted as *charge deficiency* (related to spectral flow). Then for P the Bargmann projection, they compute the index to be ± 1 by working with the explicit basis of eigenfunctions.

In [arXiv:2308.02819], we showed that $[P_X, P_Y]$ is indeed trace class, so the above integral kernel calculations directly compute its actual trace. Furthermore, as explained in this paper, the integrality and stability of this trace is extremely general, and is not limited to homogeneous P.

8 Quantum kilogram

- Large-scale index theory justified deforming true dirty experiment to idealized model, keeping σ_{Hall} invariant (thus integral). We must still do one explicit calculation (e.g. a clean Landau level) to see that it is non-zero.
- Allowed deformations are not "topology preserving" ones, but "large-scale geometry preserving". ("Topological insulator" is a misnomer.)
- For infinite-sized sample, σ_{Hall} is exactly quantized to integer multiples of $\frac{e^2}{h}$.
- Real sample is finite-sized, but *macroscopic*. So get very good approximation to exact integer of infinite-sized sample.
- So precise (10^{-10} uncertainty) that $\sigma_{\text{Hall}}^{-1}$ has been used as fundamental unit of electrical resistance, $\frac{h}{e^2}$, for some time.
- Another macroscopic experiment, *Josephson junction*, gives $\frac{h}{2e}$ =(fundamental voltage)×(second) very accurately.

  (No microscopic explanation yet, to my knowledge.)
- Both experiments got Nobel prizes. Together, they give the best available access to h (and e), via macroscopic electrical measurements.

Grand irony: Although quantum mechanical, we cannot measure h well with microscopic measurements (uncertainty principle). Instead, we see it macrosopically, e.g., blackbody radiation corrections, QHE, Josephson,...

8.1 Kilogram

The $s(econd)$ and $m(etre)$ are defined as

$$\begin{aligned} s &= 9192631770 \text{ ticks of Caesium atomic clock} \\ m &= \frac{(\text{speed of light}) \times s}{299792458} \end{aligned}$$

(Semi-)Riemannian metric on space(time) gives distances as real number × reference space/time distance unit. Speed of light converts between space unit and time unit.

Mechanics involves motion of matter, based on a property called "mass". Prior to 2019, the reference unit of mass was

$$\text{Old } kg = \text{local prototype in Paris.}$$

After QM came around (1920s to present), h has been estimated in various ways to be

$$h = 6.626\ldots??\ldots \times 10^{-34} \cdot (\text{Old } kg) \cdot m^2\, s^{-1}. \tag{45}$$

In principle, h is a universal constant, which converts space/time units to mass units. We could fix a choice for the number, and define "universal mass standard" kg_h as whatever mass balances the above equation.

However, if we don't know how to access h reliably and accurately, then we dont have access to kg_h, so such a definition would be useless in practice!

QHE is the *experimental* breakthrough which solved this problem of accurate access to h. (There is also the clever physics-engineering business of converting the "electrical signatures" of h, e to mass, by a "Watt balance", or "Kibble balance".) In 2019, kilogram was officially redefined via

$$h = 6.62607015 \times 10^{-34} \cdot (kg_h) \cdot m^2\, s^{-1}, \tag{46}$$

The numerical factor was picked to ensure that kg_h is best-known approximation to Old-kg.

Now kg_h is *quantum* by definition, and exists *forever* and *everywhere*, without deteriorating like the Old-kg!