# 扩散模型

$$\underline{dX_t = f(t) X_t \, dt} + g(t) \, dW_t$$

$$d\left(\frac{x_t}{\lambda_t}\right) = \frac{1}{\lambda_t} dX_t - \frac{d\lambda_t}{\lambda_t^2} X_t$$

需要 $\frac{1}{\lambda_t} f(t) \, dt = \frac{d\lambda_t}{\lambda_t^2}$ $\quad f(t) = \frac{d \log \lambda_t}{dt}$ $(\lambda_0 = 1)$

$L(t, X_t) = \frac{x_t}{\lambda_t}$，使用 Ito 公式

$$dL(t, X_t) = \frac{\partial L}{\partial t} dt + \frac{\partial L}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 L}{\partial x^2} g(t)^2 dt$$

$$= \frac{\partial L}{\partial t} dt + \frac{\partial L}{\partial x} f(t) X_t \, dt + \frac{\partial L}{\partial x} g(t) \, dW_t + \cancel{\frac{1}{2} \frac{\partial^2 L}{\partial x^2} g(t)^2} dt$$

$$= \frac{-\dot{\lambda}_t}{\lambda_t^2} X_t dt + \frac{f(t) X_t}{\lambda_t} dt + \frac{g(t)}{\lambda_t} dW_t$$

$$= \frac{g(t)}{\lambda_t} dW_t$$

$$\frac{X_t}{\lambda_t} = \frac{x_0}{\lambda_0} + \underbrace{\int_0^t \frac{g(\tau)}{\lambda_\tau} dW_\tau}_{\text{高斯}}$$

方差为 $\int_0^t \frac{g(\tau)^2}{\lambda_\tau^2} d\tau = \frac{\sigma_t^2}{\lambda_t^2} \Rightarrow \frac{X_t}{\lambda_t} = \frac{x_0}{\lambda_0} + \frac{\sigma_t}{\lambda_t} W$

$$X_t = \lambda_t X_0 + \sigma_t W$$

$$\frac{d}{dt}\frac{\sigma_t^2}{\lambda_t^2} = \frac{g(t)^2}{\lambda_t^2}$$

$$= \frac{1}{\lambda_t^2}\frac{d}{dt}\sigma_t^2 + \sigma_t^2 \frac{-\dot{\lambda}_t \, 2\lambda_t}{\lambda_t^4}$$

$$g(t)^2 = \frac{d}{dt}\sigma_t^2 - 2\sigma_t^2 \frac{d \log \lambda_t}{dt}$$

正向方程

$$\partial_t q_t(x) = -\nabla\cdot\left(f(t)x\, q_t(x)\right) + \frac{1}{2}\Delta\left(g(t)^2 q_t(x)\right)$$

$$\partial_t \rho_t(x) = \nabla\left(f(T-t)\, x\, \rho_t(x)\right) - \frac{1}{2}\Delta\left(g(T-t)^2 \rho_t(x)\right)$$

$$dY_t = \hat{f}(t, Y_t)\,dt + \hat{g}(t)\,dW_t$$

$$\Rightarrow$$

$$\partial_t \rho_t(x) = -\nabla\left(\hat{f}(t, x)\,\rho_t(x)\right) + \frac{1}{2}\Delta\left(\hat{g}(t)^2 \rho_t(x)\right)$$

## OU 过程

$f(t) = -1 \quad g(t) = \sqrt{2}$

$\log \lambda_t = -t \quad \Rightarrow \lambda_t = e^{-t}$

$2 = \frac{d\delta_t^2}{dt} + 2\delta_t^2 \qquad d\, 1-x = -2(1-x)\,dt$

$\Rightarrow 1-\delta_t^2 = e^{-2t}, \quad \delta_t^2 = 1-e^{-2t}$

## 分数匹配

$S_\theta(t, x)$ 　噪音 $\longrightarrow$ 图片

$$E_{q_{T-t}} \| S_\theta(t,x) - \nabla \log q_{T-t}(x) \|^2$$

$$= \iint q_{0T-t}(x \mid x_0)\, q_0(x_0) \| S_\theta(t,x) - \nabla \log q_{T-t}(x) \|^2 \, dx\, dx_0$$

$$= \iint \underline{q_{0T-t}(x \mid x_0)\, q_0(x_0)} \left[ S_\theta(t,x)^2 - \underline{2 S_\theta(t,x) \cdot \nabla \log q_{T-t}(x)} \right] dx\, dx_0$$

$$\int S_\theta(t,x)\, \nabla q_{T-t}(x)\, dx$$

$$= \int S_\theta(t,x) \int \nabla_x q_{0T-t}(x \mid x_0)\, q_0(x_0)\, dx_0\, dx$$

$$= \iint S_\theta(t,x) \int \nabla_x \log q_{0T-t}(x \mid x_0)\, \underline{q_{0T-t}(x \mid x_0)\, q_0(x_0)\, dx_0\, dx}$$

代入

$$= \iint q_{0T-t}(x \mid x_0)\, q_0(x_0) \left[ S_\theta(t,x) - \nabla_x \log q_{0T-t}(x \mid x_0) \right]^2$$

## 数值实现

$$\int_0^{T-\tau} w_{T-t}\, E_{q_0} E_{q_{0T-t}(x|x_0)} \| S_\theta(t,x) - \nabla \log q_{0T-t}(x|x_0) \|^2 \qquad T-t \to t$$

$$= \int_\tau^T w_t\, E_{q_0} E_{q_{0t}(x|x_0)} \| S_\theta(T-t, x) - \nabla \log q_{0t}(x|x_0) \|^2$$

权重： $w_t\, E_{q_0} E_{q_{0t}(x|x_0)} \| \nabla \log q_{0t}(x|x_0) \|^2$

$$= w_t\, E_{q_0} E_{q_{0t}(x|x_0)} \frac{x-m_t}{\sigma_t^2} \cdot \frac{x-m_t}{\sigma_t^2}$$

$$= w_t\, E_{q_0} \frac{d\, \sigma_t^2}{\sigma_t^4} \approx \text{常数}$$

$$w_t \sim \sigma_t^2 = 1 - e^{-2t}$$

$$= \int_\tau^T w_t\, E_{q_0} E_{q_{0t}(x|x_0)} \| S_\theta(T-t, x) + \frac{x - e^{-t} x_0}{\sigma_t^2} \|^2$$

定义 $\varepsilon_\theta(T-t, x) = -S_\theta(T-t, x) \cdot \sigma_t$

$$= \int_\tau^T w_t\, E_{q_0} E_{q_{0t}(x|x_0)} \| -\frac{\varepsilon_\theta(T-t,x)}{\sigma_t} + \frac{x - e^{-t} x_0}{\sigma_t^2} \|^2$$

$$= \int_\tau^T \frac{w_t}{\sigma_t^2} E_{q_0} E_{q_{0t}(x|x_0)} \| -\varepsilon_\theta(T-t, x) + \frac{x - e^{-t} x_0}{\sigma_t} \|^2$$

由于 $x = \lambda_t x_0 + \sigma_t W$

$$\mathbb{E}_{q_{0t}(x|x_0)} \left\| \varepsilon_\theta(T-t, x) + \frac{x - e^{-t} x_0}{\sigma_t} \right\|^2$$

$$\simeq \frac{1}{N} \sum_{i=1}^{N} \left\| \varepsilon_\theta(T-t, \lambda_t x_0 + \sigma_t W_i) - W_i \right\|^2$$