粒子滤波

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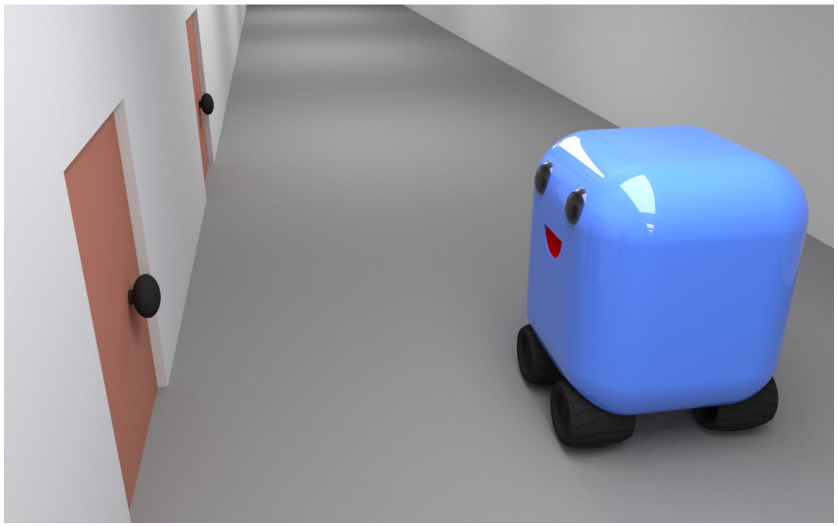
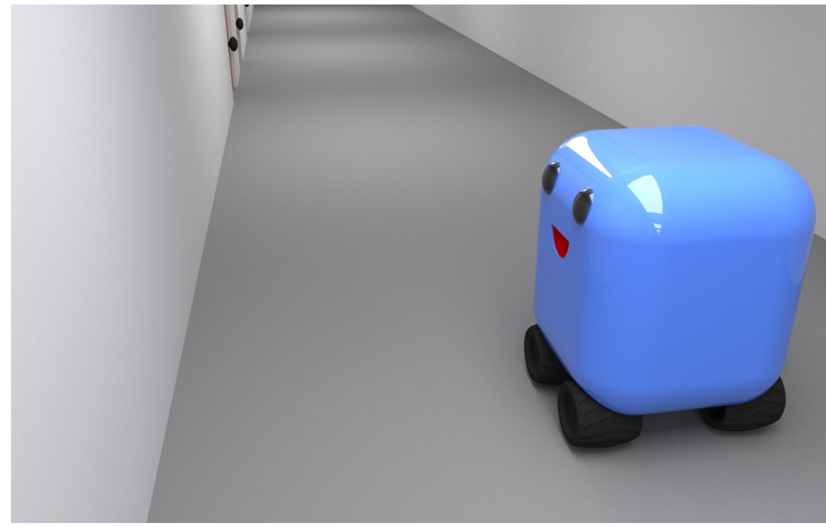
本堂课大纲

- 粒子滤波
  - 有限状态粒子滤波
  - 序贯重要性采样(SIS)粒子滤波
  - 序贯重要性重采样(SIR)粒子滤波
  - 正则化粒子滤波
  - 增广粒子滤波

数据同化问题

➢ 机器人定位问题

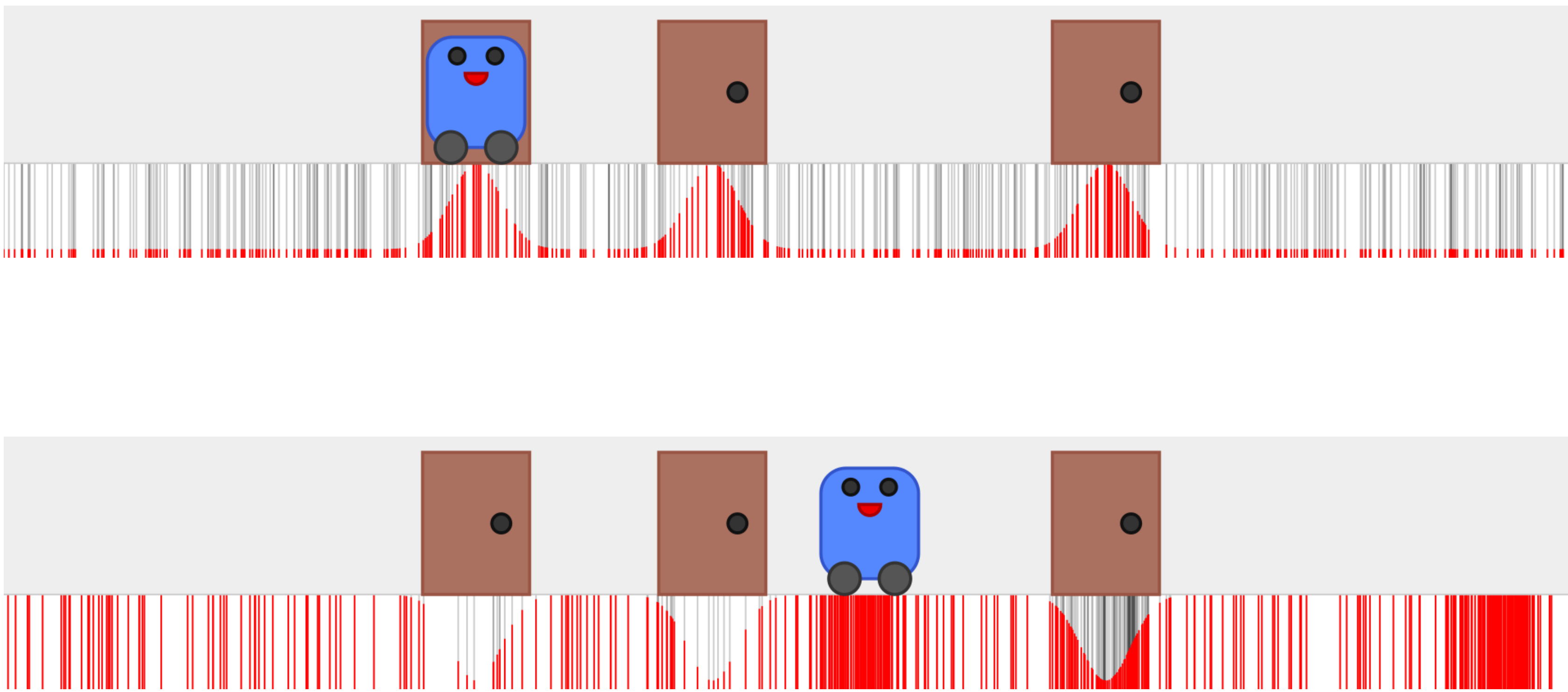




一个机器人沿着一维走廊行进，装备了一个传感器，该传感器只能判断左边是否有门。(Monte Carlo localization - Wikipedia)

数据同化问题

- 机器人定位问题



数据同化问题

- 数据同化问题

演化方程: $x_{n+1} = \mathcal{F}(x_n) + \omega_{n+1}$

观测方程: $y_{n+1} = \mathcal{H}(x_{n+1}) + \eta_{n+1}$

- 假设

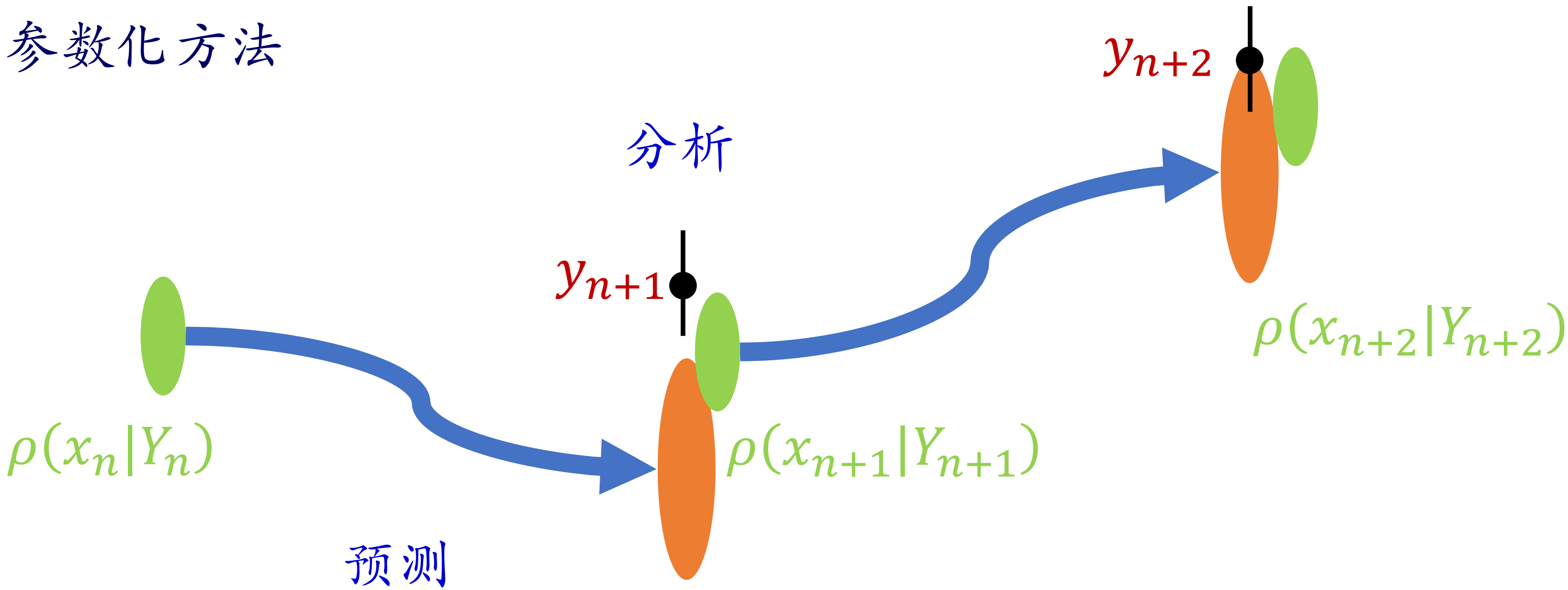
演化噪音: $\omega_{n+1} \sim \rho_\omega$

观测噪音: $\eta_{n+1} \sim \rho_\eta$

先验分布: $\rho_{\text{prior}}(x) \sim \sum_{j=1}^{J} w_0^j \delta(x - x_0^j)$

数据同化问题

➢ 参数化方法



有限状态粒子滤波

- 有限状态$x_s^1, x_s^2, \cdots, x_s^{N_s}$粒子滤波问题

n时刻后验分布：

$$x_n | Y_n \sim \sum_{j=1}^{N_s} w_n^j \delta(x - x_s^j)$$

$\mathcal{O}(N_s^2)$

预测：

$$x_{n+1} | Y_n \sim \sum_{j=1}^{N_s} \widehat{w}_{n+1}^j \delta(x - x_s^j)$$

$$\widehat{w}_{n+1}^j = \sum_{i=1}^{N_s} w_n^i \rho(x_s^i, x_s^j)$$ 演化方程的转移概率矩阵

分析：

$$x_{n+1} | Y_{n+1} \sim \sum_{j=1}^{N_s} w_{n+1}^j \delta(x - x_s^j)$$

$$\overline{w}_{n+1}^j = \widehat{w}_{n+1}^j \rho_\eta \left(y - \mathcal{H}(x_s^j)\right) \quad w_{n+1}^j = \frac{\overline{w}_{n+1}^j}{\sum_{j=1}^{J} \overline{w}_{n+1}^j}$$

序列估计问题

- 考虑采样序列服从 $\rho(x_{0:n}|Y_n)$

$$\rho(x_{0:n+1}|Y_{n+1})$$

$$=\frac{\rho(y_{n+1}|x_{0:n+1},Y_n)\rho(x_{0:n+1}|Y_n)}{\rho(y_{n+1}|Y_n)}$$

$$=\frac{\rho(y_{n+1}|Y_n,x_{0:n+1})\rho(x_{n+1}|x_{0:n},Y_n)\rho(x_{0:n}|Y_n)}{\rho(y_{n+1}|Y_n)}$$

$$=\frac{\rho(y_{n+1}|x_{n+1})\rho(x_{n+1}|x_n)\rho(x_{0:n}|Y_n)}{\rho(y_{n+1}|Y_n)}$$

序列估计问题

- 考虑采样序列服从 $\rho(x_{0:n}|Y_n)$

$$\rho(x_{0:n+1}|Y_{n+1}) = \frac{\rho(y_{n+1}|x_{n+1})\rho(x_{n+1}|x_n)\rho(x_{0:n}|Y_n)}{\rho(y_{n+1}|Y_n)}$$

边缘分布：

$$\rho(x_{n+1}|Y_{n+1}) = \frac{\rho(y_{n+1}|x_{n+1})\int\rho(x_{n+1}|x_n)\rho(x_n|Y_n)dx_n}{\rho(y_{n+1}|Y_n)}$$

预测：

$$\rho(x_{n+1}|Y_n) = \int\rho(x_{n+1}|x_n)\rho(x_n|Y_n)dx_n$$

分析：

$$\rho(x_{n+1}|Y_{n+1}) = \frac{\rho(y_{n+1}|x_{n+1})\rho(x_{n+1}|Y_n)}{\rho(y_{n+1}|Y_n)}$$

重要性采样

直接采样：

$$\rho^*(x) = \frac{1}{J}\sum_{j=1}^{J}\delta(x - x^{*j}) \qquad x^{*j} \sim \rho^*(x)$$

计算：$\mathbb{E}_{\rho^*}[f] = \int f(x)\rho^*(x)dx$

采样：$\{x^j\} \sim \rho(x)$

$$\mathbb{E}_{\rho^*}[f] \approx \rho_{\mathrm{MC}}^J(f) = \sum_{j=1}^{J} w^j f(x^j) \qquad w^j \propto \frac{\rho^*(x^j)}{\rho(x^j)}$$

重要性采样：

$$\rho^*(x) = \sum_{j=1}^{J} w^j\delta(x - x^j) \qquad x^j \sim \rho(x) \qquad w^j \propto \frac{\rho^*(x^j)}{\rho(x^j)}$$

重要性采样

- 考虑采样序列服从$\rho(x_{0:n}|Y_n)$

$$\rho(x_{0:n+1}|Y_{n+1}) = \frac{\rho(y_{n+1}|x_{n+1})\rho(x_{n+1}|x_n)\rho(x_{0:n}|Y_n)}{\rho(y_{n+1}|Y_n)}$$

采样

$$x_{0:n+1}^j \sim q(x_{0:n+1}|Y_{n+1}) \qquad w_{n+1}^j \propto \frac{\rho(x_{0:n+1}|Y_{n+1})}{q(x_{0:n+1}|Y_{n+1})}$$

那么

$$\{x_{0:n+1}^j, w_{n+1}^j\} \sim \rho(x_{0:n+1}|Y_{n+1})$$

序贯重要性采样(SIS)粒子滤波

- 序贯重要性采样(Sequential Importance Sampling)粒子滤波

假设：

$$q(x_{0:n+1}|Y_{n+1}) = q(x_{n+1}|x_{0:n}, Y_{n+1})q(x_{0:n}|Y_n)$$

重要性采样：

$$\text{生成粒子} x_{0:n}^j \sim q(x_{0:n}|Y_n), \quad w_n^j \propto \frac{p\left(x_{0:n}^j|Y_n\right)}{q\left(x_{0:n}^j|Y_n\right)}$$

采样 $x_{n+1}^j \sim q(x_{n+1}|x_{0:n}, Y_{n+1})$，计算$x_{0:n+1}^j$的权重

$$w_{n+1}^j \propto \frac{p\left(y_{n+1}|x_{n+1}^j\right)p\left(x_{n+1}^j|x_n^j\right)p\left(x_{0:n}^j|Y_n\right)}{q\left(x_{0:n+1}^j|Y_{n+1}\right)}$$

$$= \frac{p\left(y_{n+1}|x_{n+1}^j\right)p\left(x_{n+1}^j|x_n^j\right)p\left(x_{0:n}^j|Y_n\right)}{q\left(x_{n+1}^j|x_{0:n}^j, Y_{n+1}\right)q\left(x_{0:n}^j|Y_n\right)} \propto w_n^j \frac{p\left(y_{n+1}|x_{n+1}^j\right)p\left(x_{n+1}^j|x_n^j\right)}{q\left(x_{n+1}^j|x_{0:n}^j, Y_{n+1}\right)}$$

序贯重要性采样(SIS)粒子滤波

- 序贯重要性采样(Sequential Importance Sampling)粒子滤波

$$x_n|Y_n \sim \sum_{j=1}^{J} w_n^j \delta(x - x_n^j)$$

重要性采样：

$$x_{n+1}^j \sim q\left(x_{n+1}|x_{0:n}^j, Y_{n+1}\right)$$

更新权重：

$$\bar{w}_{n+1}^j = w_n^j \frac{\rho\left(y_{n+1}|x_{n+1}^j\right)\rho\left(x_{n+1}^j|x_n^j\right)}{q\left(x_{n+1}|x_{0:n}^j, Y_{n+1}\right)} \qquad w_{n+1}^j = \frac{\bar{w}_{n+1}^j}{\sum_{j=1}^{J} \bar{w}_{n+1}^j}$$

$$= w_n^j \frac{\rho_\eta\left(y_{n+1} - \mathcal{H}(x_{n+1}^j)\right)\rho_\omega\left(x_{n+1}^j - \mathcal{F}(x_n^j)\right)}{q\left(x_{n+1}|x_{0:n}^j, Y_{n+1}\right)}$$

序贯重要性采样(SIS)粒子滤波

- 序贯重要性采样(Sequential Importance Sampling)粒子滤波

$$x_n|Y_n \sim \sum_{j=1}^{J} w_n^j \delta(x - x_n^j)$$

重要性采样：

$$x_{n+1}^j \sim q(x_{n+1}|x_{0:n}^j, Y_n) := \rho(x_{n+1}|x_n^j)$$

$$x_{n+1}^j = \mathcal{F}(\hat{x}_n^j) + \omega_{n+1}^j \qquad \text{其中 } \omega_{n+1}^j \sim \rho_\omega(x)$$

更新权重：

$$\bar{w}_{n+1}^j = w_n^j \rho(y_{n+1}|x_{n+1}^j) \qquad w_{n+1}^j = \frac{\bar{w}_{n+1}^j}{\sum_{j=1}^{J} \bar{w}_{n+1}^j}$$

$$= w_n^j \rho_\eta\left(y_{n+1} - \mathcal{H}(x_{n+1}^j)\right)$$

权重退化为0

序贯重要性采样(SIS)粒子滤波

- 有效粒子数(Effective sample size)

$$\widehat{N_{eff}} = \frac{1}{\sum_{j=1}^{J}(w_{n+1}^{j})^2}$$

$$1 \leq \widehat{N_{eff}} \leq J$$

序贯重要性重采样(SIR)粒子滤波

- 序贯重要性重采样(Sequential Importance Resampling)粒子滤波

$$x_n|Y_n \sim \sum_{j=1}^{J} w_n^j \delta(x - x_n^j)$$

重采样: $\hat{x}_n^j \sim \sum_{j=1}^{J} w_n^j \delta(x - x_n^j)$ 生成 $\alpha \sim \text{Uniform}[0,1]$

$$x_n|Y_n \sim \frac{1}{J}\sum_{j=1}^{J} \delta(x - \hat{x}_n^j)$$

重要性采样： $x_{n+1}^j \sim \rho\left(x_{n+1}|\hat{x}_n^j\right)$

$$x_{n+1}^j = \mathcal{F}\left(\hat{x}_n^j\right) + \omega_{n+1}^j \qquad \text{其中 } \omega_{n+1}^j \sim \rho_\omega(x)$$

更新权重：

$$\overline{w}_{n+1}^j = \frac{1}{J}\rho_\eta\left(y_{n+1} - \mathcal{H}\left(x_{n+1}^j\right)\right)$$

$$w_{n+1}^j = \frac{\overline{w}_{n+1}^j}{\sum_{j=1}^{J} \overline{w}_{n+1}^j}$$

序贯重要性重采样(SIR)粒子滤波

- 算子理解

$$x_n|Y_n \sim \sum_{j=1}^{J} w_n^j \delta(x - x_n^j)$$

$$\rho(x_{n+1}|Y_{n+1})$$

$$= \frac{\rho(y_{n+1}|x_{n+1},Y_n)\rho(x_{n+1}|Y_n)}{\rho(y_{n+1}|Y_n)}$$

$$\propto \rho(y_{n+1}|x_{n+1},Y_n)\int \rho(x_{n+1}|x_n,Y_n)\rho(x_n|Y_n)dx_n$$

$$\approx \rho(y_{n+1}|x_{n+1},Y_n)\sum_j w_n^j \rho\left(x_{n+1}\middle|x_n^j,Y_n\right)$$

预测： $x_{n+1}^j, \frac{1}{J} \sim \rho(x_{n+1}|Y_n) \approx \sum_j w_n^j \rho\left(x_{n+1}\middle|x_n^j,Y_n\right)$

分析： $w_{n+1}^j \propto \frac{1}{J}\rho\left(y_{n+1}\middle|x_{n+1}^j,Y_n\right)$

序贯重要性重采样(SIR)粒子滤波

- 算子理解

$$\mathcal{P}\rho(y) = \int \rho_\omega\big(y - \mathcal{F}(x)\big)\rho(x)\,dx$$

$$\mathcal{A}_n\rho(x) = \rho(x|\mathcal{H}(x) + \eta = y_n)$$

$$= \frac{\rho_\eta\big(y_n - \mathcal{H}(x)\big)\rho(x)}{\int \rho_\eta\big(y_n - \mathcal{H}(x)\big)\rho(x)\,dx}$$

$$\mathcal{S}^J\rho(x) = \frac{1}{J}\sum_{j=1}^{J}\delta\big(x - x^j\big) \qquad x^j \sim \rho$$

序贯重要性重采样(SIR)粒子滤波

- 滤波分布

$$\rho_n^*(x_n) = \rho(x_n|Y_n)$$

$$\rho_{n+1}^* = \mathcal{A}_{n+1}\mathcal{P}\rho_n^*$$

- SIR近似分布

$$\rho_n^J(x) = \sum_{j=1}^{J} w_n \delta(x - x_n^j)$$

$$\rho_{n+1}^J = \mathcal{A}_{n+1}\mathcal{S}^J\mathcal{P}\rho_n^J$$

序贯重要性重采样(SIR)粒子滤波

重要性重采样粒子滤波器收敛性

假设存在 $\kappa \in (0,1)$，对于$n = 1,2,\cdots,N$

$$\kappa \le \rho_\eta\big(y_n - \mathcal{H}(x)\big) \le 1/\kappa$$

并且$\rho_0^J = \rho_0^*$。对于滤波分布ρ_n^*，我们的近似ρ_n^J满足

$$d(\rho_n^*, \rho_n^J) = \sup_{|f|_\infty \le 1} \left(\mathbb{E}\left[\left(\rho_n^*(f) - \rho_n^J(f)\right)^2\right]\right)^{1/2} \le \frac{c(n,\kappa)}{\sqrt{J}}$$

其中J是粒子数，$\rho(f) = \int f\rho\, dx, \mathbb{E}$是关于$S^J$重采样的期望，$c(n,\kappa) = \frac{\lambda(\lambda^n - 1)}{\lambda - 1}$，$\lambda = 2/\kappa^2$。

粒子滤波

- 粒子坍缩(Particle collapse)

$$x_n|Y_n \sim \sum_{j=1}^{J} w_n \delta(x - x_n^j)$$

重采样: $\hat{x}_n^j \sim \rho_n^J(x)$ 生成 $\alpha \sim \text{Uniform}[0,1]$

$$x_n|Y_n \sim \frac{1}{J}\sum_{j=1}^{J} \delta(x - \hat{x}_n^j)$$

$$x_{n+1}^j = \mathcal{F}\left(\hat{x}_n^j\right) + \omega_{n+1}^j \qquad \text{其中 } \omega_{n+1}^j \sim \rho_\omega(x)$$

当J不大，演化噪音ω_{n+1}^j很小时，若$w_1 \gg w_i$,发生粒子坍缩。

正则化(Regularized)粒子滤波

- 重采样(额外添加噪音)

$$x_n|Y_n \sim \sum_{j=1}^{J} w_n^j K(x - x_n^j)$$

核函数

$$\int K(x)dx = 1$$

有平移不变性的核函数

$$K(x) = K(\| x \|)$$

增广(Auxiliary)粒子滤波

- 算子理解

$$x_n|Y_n \sim \sum_{j=1}^{J} w_n^j \delta(x - x_n^j)$$

$$\rho(x_{n+1}|Y_{n+1})$$

$$= \frac{\rho(y_{n+1}|x_{n+1},Y_n)\rho(x_{n+1}|Y_n)}{\rho(y_{n+1}|Y_n)}$$

$$\propto \rho(y_{n+1}|x_{n+1},Y_n)\int \rho(x_{n+1}|x_n,Y_n)\rho(x_n|Y_n)dx_n$$

$$\approx \rho(y_{n+1}|x_{n+1},Y_n)\sum_j w_n^j \rho\left(x_{n+1}\middle|x_n^j,Y_n\right)$$

$$\mathcal{O}(N_s)$$

预测： $x_{n+1}^j \sim g(x_{n+1}|Y_{n+1})$

分析： $w_{n+1}^j \propto \dfrac{\rho\left(y_{n+1}\middle|x_{n+1}^j,Y_n\right)\rho\left(x_{n+1}^j\middle|Y_n\right)}{g\left(x_{n+1}^j\middle|Y_{n+1}\right)}$

增广(Auxiliary)粒子滤波

- 算子理解

$$x_n|Y_n \sim \sum_{j=1}^{J} w_n^j \delta(x - x_n^j)$$

$$\rho(x_{n+1}|Y_{n+1}) \propto \rho(y_{n+1}|x_{n+1}, Y_n) \sum_j w_n^j \rho(x_{n+1}|x_n^j, Y_n)$$

$$\rho(x_{n+1}, k|Y_{n+1}) \propto \rho(y_{n+1}|x_{n+1}, Y_n) w_n^k \rho(x_{n+1}|x_n^k, Y_n)$$

预测： $x_{n+1}^j, i^j \sim g(x_{n+1}, k|Y_{n+1})$

分析： $w_{n+1}^j \propto \dfrac{\rho\left(y_{n+1}\middle|x_{n+1}^j, Y_n\right) w_n^{i^j} \rho\left(x_{n+1}^j\middle|x_n^{i^j}\right)}{g\left(x_{n+1}^j, i^j\middle|Y_{n+1}\right)}$

$$x_{n+1}^j, i^j, w_{n+1}^j \sim \rho(y_{n+1}|x_{n+1}, Y_n) w_n^k \rho(x_{n+1}|x_n^k, Y_n)$$

$$x_{n+1}^j, w_{n+1}^j \sim \rho(y_{n+1}|x_{n+1}, Y_n)\rho(x_{n+1}|Y_n)$$

增广(Auxiliary)粒子滤波

$$x_n|Y_n \sim \sum_{j=1}^{J} w_n^j \delta(x - x_n^j)$$

$$\rho(x_{n+1}, k|Y_{n+1}) \propto \rho(y_{n+1}|x_{n+1}, Y_n) w_n^k \rho\left(x_{n+1}\middle|x_n^k, Y_n\right)$$

$$g(x_{n+1}, k|Y_{n+1}) \propto \rho\left(y_{n+1}\middle|\mu_{n+1}^k\right) w_n^k \rho\left(x_{n+1}\middle|x_n^k\right)$$

μ_{n+1}^k是 $x_{n+1}|x_n^k$ 的期望或任意一个采样等

预测： $x_{n+1}^j, i^j \sim g(x_{n+1}, k|Y_{n+1})$

分析： $w_{n+1}^j \propto \dfrac{\rho\left(y_{n+1}\middle|x_{n+1}^j, Y_n\right) w_n^{i^j} \rho\left(x_{n+1}^j\middle|x_n^{i^j}\right)}{g\left(x_{n+1}^j, i^j\middle|Y_{n+1}\right)}$

总结

- 粒子滤波
  - 基于序贯重要性采样的粒子滤波
  - 基于序贯重要性重采样的粒子滤波
  - 当 $J \to \infty$ ，收敛到滤波分布
  - 潜在问题：运算量大、不稳定、粒子坍缩
  - 改进：正则化、增广粒子滤波

扩展阅读

- 粒子滤波

使用综述: Arulampalam, M. Sanjeev, et al. "A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking." *IEEE Transactions on signal processing* 50.2 (2002): 174-188.

使用综述: Elfring, Jos, Elena Torta, and René Van De Molengraft. "Particle filters: A hands-on tutorial." *Sensors* 21.2 (2021): 438..

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贝叶斯采样： Del Moral, Pierre, Arnaud Doucet, and Ajay Jasra. "Sequential monte carlo samplers." Journal of the Royal Statistical Society Series B: Statistical Methodology 68.3 (2006): 411-436.