1. 算子理解

我们定义如下算子

$$
\begin{aligned}
&\mathcal{P}\rho(y)=\int \rho_\omega(y-\mathcal{F}(x))\rho(x)dx \\
&\mathcal{A}_j\rho(x)=\rho(x|\mathcal{H}(x)+\eta_j=y_j)=\frac{\rho_\eta(y_j-\mathcal{H}(x))\rho(x)}{\int \rho_\eta(y_j-\mathcal{H}(x))\rho(x)dx} \\
&S^J\rho(x)=\frac{1}{J}\sum_{j=1}^{J}\delta(x-x^j) \qquad \text{其中} x^j\sim\rho
\end{aligned}
$$

证明

对于滤波分布，我们有

$$
\begin{aligned}
\rho^*_{n+1}=\rho(x_{n+1}|Y_{n+1})=\rho(x_{n+1}|y_{n+1},Y_n)&=\frac{\rho(y_{n+1}|x_{n+1},Y_n)\rho(x_{n+1}|Y_n)}{\int \rho(y_{n+1}|x_{n+1},Y_n)\rho(y_{n+1}|x_{n+1},Y_n)dx_{n+1}} \\
&=\frac{\rho(y_{n+1}|x_{n+1},Y_n)\rho(x_{n+1}|Y_n)}{\int \rho(y_{n+1}|x_{n+1},Y_n)\rho(x_{n+1}|Y_n)dx_{n+1}} \\
&=\frac{\rho(y_{n+1}|x_{n+1},Y_n)\int \rho(x_{n+1},x_n|Y_n)\rho(x_n|Y_n)dx_n}{\int \rho(y_{n+1}|x_{n+1},Y_n)\rho(x_{n+1}|Y_n)dx_{n+1}} \\
&=\frac{\rho(y_{n+1}|x_{n+1},Y_n)\mathcal{P}\rho(x_n|Y_n)}{\int \rho(y_{n+1}|x_{n+1},Y_n)\rho(x_{n+1}|Y_n)dx_{n+1}} \\
&=\mathcal{A}_{n+1}\mathcal{P}\rho(x_n|Y_n)
\end{aligned}
$$

对于我们的近似的预测和分析，我们有

$$\rho_n^J(x)=\sum_{j=1}^J w_n\delta(x-x_n^j)$$

$$\mathcal{P}\rho_n^J(x)=\sum_{j=1}^J w_n\rho_\omega(x-\mathcal{F}(x_n^j))$$

采样 $\hat{x}_n^j\sim\rho_n^J \quad \omega_n^j\sim\rho_\omega$ ， 那么 $x_{n+1}^j=\mathcal{F}(\hat{x}_n^j)+\omega_n^j\sim\mathcal{P}\rho_n^J(x)$

$$S^J\mathcal{P}\rho_n^J(x)=\frac{1}{J}\sum_{j=1}^J\delta(x-x_{n+1}^j)$$

$$\mathcal{A}_{n+1}S^J\mathcal{P}\rho_n^J(x)\propto\frac{1}{J}\sum_{j=1}^J\rho_\eta(y_{n+1}-\mathcal{H}(x))\delta(x-x_{n+1}^j)=\frac{1}{J}\sum_{j=1}^J\rho_\eta(y_{n+1}-\mathcal{H}(x_{n+1}^j))\delta(x-x_{n+1}^j)\propto\rho_{n+1}^J(x)$$

2.引理

假设存在 $\kappa\in(0,1)$，对于$n=1,2,\cdots,N$

$$\kappa\le\rho_\eta(y_n-\mathcal{H}(x))\le\kappa^{-1}$$

我们有

$$d(\mathcal{A}_n\rho,\mathcal{A}_n\rho')\le\frac{2}{\kappa^2}d(\rho,\rho')\qquad d(\mathcal{P}\rho,\mathcal{P}\rho')\le d(\rho,\rho')\qquad d(\rho,\mathcal{S}^J\rho)\le\frac{1}{\sqrt{J}}$$

其中

$$d(\rho,\rho')^2=\sup_{|f|_\infty\le1}\mathbb{E}_{x_n^j\sim\rho_n^*}[\big(\rho(f)-\rho'(f)\big)^2]\qquad\rho f=\int\rho f dx$$

证明

对于$\mathcal{A}_n$， 我们定义 $l(x)=\rho_\eta(y_n-\mathcal{H}(x))$

$$
\begin{aligned}
|\mathcal{A}_n\rho(f)-\mathcal{A}_n\rho'(f)| &= |\frac{\rho(fl)}{\rho(l)}-\frac{\rho'(fl)}{\rho'(l)}| \\
&= |\frac{\rho(fl)-\rho'(fl)}{\rho(l)}+\rho'(fl)(\frac{1}{\rho(l)}-\frac{1}{\rho'(l)})| \\
&\leq |\frac{\rho(fl)-\rho'(fl)}{\rho(l)}|+|\frac{\rho'(fl)}{\rho'(l)}||\frac{\rho'(l)-\rho(l)}{\rho(l)}| \qquad \text{使用} \qquad |f|_\infty \leq 1 \\
&\leq |\frac{\rho(fl)-\rho'(fl)}{\rho(l)}|+|\frac{\rho'(l)-\rho(l)}{\rho(l)}| \\
&\leq \frac{1}{\kappa}\Big(|\rho(fl)-\rho'(fl)|+|\rho'(l)-\rho(l)|\Big)
\end{aligned}
$$

我们有

$$
\begin{aligned}
|\mathcal{A}_n\rho(f)-\mathcal{A}_n\rho'(f)|^2 &\leq \frac{2}{\kappa^2}\Big(|\rho(fl)-\rho'(fl)|^2+|\rho'(l)-\rho(l)|^2\Big) \\
&\leq \frac{2}{\kappa^4}\Big(|\rho(f\kappa l)-\rho'(f\kappa l)|^2+|\rho'(\kappa l)-\rho(\kappa l)|^2\Big) \\
&\leq \frac{2}{\kappa^4}\Big(|\rho(f\kappa l)-\rho'(f\kappa l)|^2+|\rho'(\kappa l)-\rho(\kappa l)|^2\Big)
\end{aligned}
$$

由于

$$
\sup_{|f|_\infty\leq 1}\mathbb{E}|\rho(f\kappa l)-\rho'(f\kappa l)|^2 \geq \mathbb{E}|\rho(\kappa l)-\rho'(\kappa l)|^2
$$

由于 $|\kappa l| \leq 1$

$$
\sup_{|f|_\infty\leq 1}\mathbb{E}|\rho(f\kappa l)-\rho'(f\kappa l)|^2 \leq \sup_{|f|_\infty\leq 1}\mathbb{E}|\rho(f)-\rho'(f)|^2
$$

因此

$$
\begin{aligned}
\sup_{|f|_\infty \le 1} \mathbb{E}|\mathcal{A}_n\rho(f) - \mathcal{A}_n\rho'(f)|^2 &\le \frac{2}{\kappa^4} \sup_{|f|_\infty \le 1} \mathbb{E}\Big(|\rho(f\kappa l) - \rho'(f\kappa l)|^2 + |\rho'(\kappa l) - \rho(\kappa l)|^2\Big) \\
&\le \frac{4}{\kappa^4} \sup_{|f|_\infty \le 1} \mathbb{E}|\rho(f) - \rho'(f)|^2
\end{aligned}
$$

对于$\mathcal{P}$，我们定义 $q(x) = \int \rho_\omega(y - \mathcal{F}(x))f(y)dy$

$$
\mathcal{P}\rho(f) = \int\int \rho_\omega(y - \mathcal{F}(x))\rho(x)dxf(y)dy = \int\int \rho_\omega(y - \mathcal{F}(x))f(y)dy\rho(x)dx := \int q(x)\rho(x)dx
$$

由于 $|f|_\infty \le 1$，我们有$|q|_\infty \le 1$。因此

$$
\begin{aligned}
&|\mathcal{P}\rho(f) - \mathcal{P}\rho'(f)| = |\rho(q) - \rho'(q)| \\
&\sup_{|f|_\infty \le 1} \mathbb{E}|\mathcal{P}\rho(f) - \mathcal{P}\rho'(f)|^2 \le \sup_{|f|_\infty \le 1} \mathbb{E}|\rho(q) - \rho'(q)|^2 \le \sup_{|f|_\infty \le 1} \mathbb{E}|\rho(f) - \rho'(f)|^2
\end{aligned}
$$

对于$\mathcal{S}^J$，我们有

$$
\begin{aligned}
d(\rho, \mathcal{S}^J\rho)^2 = \mathbb{E}\Big[\big(\frac{1}{J}\sum_{j=1}^{J} f(\theta^j) - \mathbb{E}_\rho[f]\big)^2\Big] &= \mathbb{E}\Big[\frac{1}{J}\sum_{j=1}^{J}\big(f(\theta^j) - \mathbb{E}[f]\big)^2\Big] \\
&= \frac{1}{J}\mathbb{E}\Big[\big(f(\theta) - \mathbb{E}[f]\big)^2\Big] \\
&= \frac{1}{J}\Big(\mathbb{E}[f^2] - (\mathbb{E}[f])^2\Big) \\
&\le \frac{1}{J}\mathbb{E}[f^2] \\
&\le \frac{1}{J}
\end{aligned}
$$

3. 粒子滤波

假设存在 $\kappa \in (0,1)$，对于$n = 1, 2, \cdots, N$

$$\kappa \le \rho_\eta(y_n - \mathcal{H}(x)) \le \kappa^{-1}$$

并且 $\rho_0^J = \rho_0^*$。对于滤波分布ρ_n^*，我们的近似ρ_n^J满足

$$d(\rho_n^*, \rho_n^J) \le \frac{c(J, \kappa)}{\sqrt{J}}$$

其中J是粒子数，c是不依赖于N的常数。

定理证明

我们定义滤波分布以及我们的近似

$$\rho_n^*(x_n) = \rho^*(x_n|Y_n) \qquad \rho_n^J(x_n) = \rho(x_n|Y_n)$$

我们定义 $e_n = d(\rho_n^*, \rho_n^J)$，使用三角不等式，可以得到

$$\begin{aligned}
e_{n+1} &= d(\rho_{n+1}^*, \rho_{n+1}^J) = d(\mathcal{A}_{n+1}\mathcal{P}\rho_n^*, \mathcal{A}_{n+1}S^J\mathcal{P}\rho_n^J(x)) \\
&\le d(\mathcal{A}_{n+1}\mathcal{P}\rho_n^*, \mathcal{A}_{n+1}\mathcal{P}\rho_n^J(x)) + d(\mathcal{A}_{n+1}\mathcal{P}\rho_n^J(x), \mathcal{A}_{n+1}S^J\mathcal{P}\rho_n^J(x)) \\
&\le \frac{2}{\kappa^2}\left[d(\mathcal{P}\rho_n^*, \mathcal{P}\rho_n^J(x)) + d(\mathcal{P}\rho_n^J(x), S^J\mathcal{P}\rho_n^J(x))\right] \\
&\le \frac{2}{\kappa^2}\left[d(\rho_n^*, \rho_n^J(x)) + \frac{1}{\sqrt{J}}\right] \\
&= \frac{2}{\kappa^2}\left[e_n + \frac{1}{\sqrt{J}}\right]
\end{aligned}$$

定义$\lambda = 2/\kappa^2$使用归纳法，我们有

$$e_n = \lambda^n e_0 + \frac{\lambda(1-\lambda^n)/(1-\lambda)}{\sqrt{J}} = \frac{\lambda(1-\lambda^n)/(1-\lambda)}{\sqrt{J}}$$

In [ ]: