1. 卡尔曼滤波算法

对于线性数据同化问题

$$
\begin{aligned}
\text{演化方程:}\quad & x_{n+1} = Fx_n + \omega_{n+1}, \qquad \omega_{n+1} \sim \mathcal{N}(0, \Sigma_\omega) \qquad (1)\\
\text{观测方程:}\quad & y_{n+1} = Gx_{n+1} + \eta_{n+1}, \qquad \eta_{n+1} \sim \mathcal{N}(0, \Sigma_\eta) \qquad (2)
\end{aligned}
$$

定义 $Y_n = \{y_\ell\}_{\ell=1}^n$, 卡尔曼滤波算法分为预测步骤和分析步骤近似，分别近似预测分布 $\hat{\rho}_n(x_n) = \rho(x_n|Y_{n-1})$ 和滤波分布 $\rho_n(x_n) := \rho(x_n|Y_n)$。

它们满足如下迭代关系

$$
\begin{aligned}
\hat{\rho}_{n+1}(x_{n+1}) &= \mathcal{N}(x; \hat{m}_{n+1}, \hat{C}_{n+1}) \qquad \hat{m}_{n+1} = Fm_n \qquad \hat{C}_{n+1} = FC_nF^T + \Sigma_\omega\\
\rho_{n+1}(x_{n+1}) &= \mathcal{N}(x; m_{n+1}, C_{n+1}) \qquad m_{n+1} = \hat{m}_{n+1} + K_{n+1}d_{n+1} \qquad C_{n+1} = (I - K_{n+1}H)\hat{C}_{n+1}
\end{aligned}
$$

其中

$$
d_{n+1} = y_{n+1} - H\hat{m}_{n+1} \qquad S_{n+1} = H\hat{C}_{n+1}H^T + \Sigma_\eta \qquad K_{n+1} \qquad = \hat{C}_{n+1}H^TS_{n+1}^{-1}
$$

对于演化步骤

$$
\begin{aligned}
\hat{\rho}_{n+1}(x_{n+1}) = \rho(x_{n+1}|Y_n) &= \int \rho(x_{n+1}|x_n, Y_n)\rho(x_n|Y_n)dx_n = \int \rho(x_{n+1}|x_n)\rho(x_n|Y_n)dx_n\\
&= \int \mathcal{N}(x_{n+1}; Fx_n, \Sigma_\omega)\mathcal{N}(x_n; m_n, C_n)dx_n\\
&\propto \int \exp\Big(-\frac{1}{2}(x_{n+1} - Fx_n)^T\Sigma_\omega^{-1}(x_{n+1} - Fx_n) - \frac{1}{2}(x_n - m_n)^TC_n^{-1}(x_n - m_n)\Big)dx_n\\
&\propto \int \exp\Big(-\frac{1}{2}x_n^T\big(F^T\Sigma_\omega^{-1}F + C_n^{-1}\big)x_n - \frac{1}{2}x_n^T - m_n)^TC_n^{-1}(x_n - m_n)\Big)dx_n
\end{aligned}
$$

证明

我们采用归纳法证明， 根据先验，我们有 $\rho_0(x_0) = \mathcal{N}(x; m_0, C_0)$

对于演化步骤

$$
\begin{aligned}
\hat{\rho}_{n+1}(x_{n+1}) = \rho(x_{n+1}|Y_n) &= \int \rho(x_{n+1}|x_n, Y_n)\rho(x_n|Y_n)dx_n = \int \rho(x_{n+1}|x_n)\rho(x_n|Y_n)dx_n\\
&= \int \mathcal{N}(x_{n+1}; Fx_n, \Sigma_\omega)\mathcal{N}(x_n; m_n, C_n)dx_n\\
&\propto \int \exp\Big(-\frac{1}{2}(x_{n+1} - Fx_n)^T\Sigma_\omega^{-1}(x_{n+1} - Fx_n) - \frac{1}{2}(x_n - m_n)^TC_n^{-1}(x_n - m_n)\Big)dx_n\\
&\propto \int \exp\Big(-\frac{1}{2}x_n^T\big(F^T\Sigma_\omega^{-1}F + C_n^{-1}\big)x_n + x_n^T\big(C_n^{-1}m_n + F^T\Sigma_\omega^{-1}x_{n+1}\big) - \frac{1}{2}x_{n+1}^T\Sigma_\omega^{-1}x_{n+1}\Big)dx_n\\
&\propto \int \exp\Big(-\frac{1}{2}(x_n - r_n)^T\big(F^T\Sigma_\omega^{-1}F + C_n^{-1}\big)(x_n - r_n) + \frac{1}{2}r_n^T\big(F^T\Sigma_\omega^{-1}F + C_n^{-1}\big)r_n - \frac{1}{2}x_{n+1}^T\Sigma_\omega^{-1}x_{n+1}\Big)dx_n\\
&\text{使用} \qquad r_n = \big(F^T\Sigma_\omega^{-1}F + C_n^{-1}\big)^{-1}\big(C_n^{-1}m_n + F^T\Sigma_\omega^{-1}x_{n+1}\big)\\
&\propto \exp\frac{1}{2}\Big(\big(C_n^{-1}m_n + F^T\Sigma_\omega^{-1}x_{n+1}\big)^T\big(F^T\Sigma_\omega^{-1}F + C_n^{-1}\big)^{-1}\big(C_n^{-1}m_n + F^T\Sigma_\omega^{-1}x_{n+1}\big) - \frac{1}{2}x_{n+1}^T\Sigma_\omega^{-1}x_{n+1}\Big)\\
&\propto \exp\Big(-\frac{1}{2}x_{n+1}^T\big(\Sigma_\omega^{-1} - \Sigma_\omega^{-1}F\big(F^T\Sigma_\omega^{-1}F + C_n^{-1}\big)^{-1}F\Sigma_\omega^{-1}\big)x_{n+1} + x_{n+1}^T\Sigma_\omega^{-1}F\big(F^T\Sigma_\omega^{-1}F + C_n^{-1}\big)^{-1}C_n^{-1}m_n\Big)\\
&\propto \exp\Big(-\frac{1}{2}(x_{n+1} - Fm_n)^T\big(F^TC_nF + \Sigma_\omega\big)^{-1}(x_{n+1} - Fm_n)\Big)\\
&\propto \mathcal{N}(x_{n+1}; Fm_n, F^TC_nF + \Sigma_\omega)
\end{aligned}
$$

另一种思考方式

$$
\begin{aligned}
\hat{\rho}_{n+1}(x_{n+1}) &= \rho(x_{n+1}|Y_n) \sim \rho(Fx_n + \omega_{n+1}|Y_n)\\
\mathbb{E}[Fx_n + \omega_{n+1}|Y_n] &= F\mathbb{E}[x_n|Y_n] + \mathbb{E}[\omega_{n+1}|Y_n] = Fm_n\\
\mathrm{Cov}[Fx_n + \omega_{n+1}|Y_n] &= \mathbb{E}[(Fx_n + \omega_{n+1} - Fm_n)(Fx_n + \omega_{n+1} - Fm_n)^T|Y_n]\\
&= \mathbb{E}[(Fx_n - Fm_n)(Fx_n - Fm_n)^T|Y_n] + \mathbb{E}[\omega_{n+1}\omega_{n+1}^T|Y_n]\\
&= FC_nF^T + \Sigma_\omega
\end{aligned}
$$

对于分析步骤， 我们使用贝叶斯法则

$$
\begin{aligned}
\rho_{n+1}(x_{n+1}) = \rho(x_{n+1}|Y_{n+1}) = \rho(x_{n+1}|Y_n, y_{n+1}) &= \frac{\rho(y_{n+1}|x_{n+1}, Y_n)\rho(x_{n+1}|Y_n)}{\rho(y_{n+1}|Y_n)}\\
&\propto \rho(y_{n+1}|x_{n+1}, Y_n)\rho(x_{n+1}|Y_n) \propto \rho(y_{n+1}|x_{n+1})\rho(x_{n+1}|Y_n)\\
&\propto \mathcal{N}(y_{n+1}; Hx_{n+1}, \Sigma_\eta)\mathcal{N}(x_{n+1}; \hat{m}_{n+1}, \hat{C}_{n+1})\\
&\propto \exp\Big(-\frac{1}{2}(y_{n+1} - Hx_{n+1})^T\Sigma_\eta^{-1}(y_{n+1} - Hx_{n+1}) - \frac{1}{2}(x_{n+1} - \hat{m}_{n+1})\hat{C}_{n+1}^{-1}(x_{n+1} - \hat{m}_{n+1})\Big)\\
&\propto \mathcal{N}(x_{n+1}; m_{n+1}, C_{n+1})
\end{aligned}
$$

其中

$$
C_{n+1} = \big(H^T\Sigma_\eta^{-1}H + \hat{C}_{n+1}^{-1}\big)^{-1} \qquad m_{n+1} = \big(H^T\Sigma_\eta^{-1}H + \hat{C}_{n+1}^{-1}\big)^{-1}\big(H^T\Sigma_\eta^{-1}y_{n+1} + \hat{C}_{n+1}^{-1}\hat{m}_{n+1}\big)
$$

我们可以进一步简化

$$
\begin{aligned}
m_{n+1} &= \hat{m}_{n+1} + \hat{C}_{n+1}H^T(\Sigma_\eta + H\hat{C}_{n+1}H^T)^{-1}(y_{n+1} - H\hat{m}_{n+1})\\
C_{n+1} &= \hat{C}_{n+1} - \hat{C}_{n+1}H^T(\Sigma_\eta + H\hat{C}_{n+1}H^T)^{-1}H\hat{C}_{n+1}
\end{aligned}
$$

因此我们可以定义

$$
d_{n+1} = y_{n+1} - H\hat{m}_{n+1} \qquad S_{n+1} = H\hat{C}_{n+1}H^T + \Sigma_\eta \qquad K_{n+1} \qquad = \hat{C}_{n+1}H^TS_{n+1}^{-1}
$$

2. 卡尔曼滤波的最优性

对于任意序列 $\{z_n\}$，我们有

$$
\mathbb{E}[|x_n - m_n|^2|Y_n] \leq \mathbb{E}[|x_n - z_n|^2|Y_n]
$$

证明

$$
\begin{aligned}
\mathbb{E}[|x_n - z_n|^2|Y_n] &= \mathbb{E}[|x_n - m_n + m_n - z_n|^2|Y_n]\\
&= \mathbb{E}[|x_n - m_n|^2 + |m_n - z_n|^2 + 2 < x_n - m_n, m_n - z_n > |Y_n]\\
&= \mathbb{E}[|x_n - m_n|^2 + |m_n - z_n|^2|Y_n]\\
&\geq \mathbb{E}[|x_n - m_n|^2|Y_n]
\end{aligned}
$$

3. 卡尔曼平滑算法

对于贝叶斯平滑问题，我们有

$$
\rho_{\text{post}}(X;Y) = \frac{1}{Z}e^{-\Phi(X,Y)}\rho_{\text{prior}}(X) \propto e^{-\Phi_R(X,Y)}
$$

其中

$$
\begin{aligned}
\Phi_R &= \frac{1}{2}\sum_{n=0}^{N-1}\|\Sigma_\eta^{-1/2}(y_{n+1} - Hx_{n+1})\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\|\Sigma_\omega^{-1/2}(x_{n+1} - Fx_n)\|^2 + \frac{1}{2}\|C_0^{-1/2}(x_0 - m_0))\|^2\\
&= \frac{1}{2}(x - m_{\text{post}})^TC_{\text{post}}^{-1}(x - m_{\text{post}})
\end{aligned}
$$

我们有

$$
\nabla_x\Phi_R = C_{\text{post}}^{-1}(x - m_{\text{post}})
$$

由于

$$
\begin{aligned}
&\nabla_{x_0}\Phi_R = -F^T\Sigma_\omega^{-1}(x_1 - Fx_0) + C_0^{-1}(x_0 - m_0)\\
&\nabla_{x_n}\Phi_R = -H^T\Sigma_\eta^{-1}(y_n - Hx_n) - F^T\Sigma_\omega^{-1}(x_{n+1} - Fx_n) + \Sigma_\omega^{-1}(x_n - Fx_{n-1}) \qquad 0 < n < N\\
&\nabla_{x_N}\Phi_R = -H^T\Sigma_\eta^{-1}(y_N - Hx_N) + \Sigma_\omega^{-1}(x_N - Fx_{N-1})
\end{aligned}
$$

定义 $r = C_{\text{post}}^{-1}m_{\text{post}}$， r 满足

$$
\begin{aligned}
r_0 &= C_0^{-1}m_0\\
r_n &= -H^T\Sigma_\eta^{-1}y_n \qquad 0 < n \leq N
\end{aligned}
$$

我们得到 $\Omega = C_{\text{post}}^{-1}$ 是分块三对角矩阵

$$
\begin{aligned}
&\nabla_{x_0}\nabla_{x_0}\Phi_R = F^T\Sigma_\omega^{-1}F + C_0^{-1} \qquad \nabla_{x_0}\nabla_{x_1}\Phi_R = -F^T\Sigma_\omega^{-1}\\
&\nabla_{x_n}\nabla_{x_{n+1}}\Phi_R = -F^T\Sigma_\omega^{-1} \qquad \nabla_{x_n}\nabla_{x_n}\Phi_R = H^T\Sigma_\eta^{-1}H + F^T\Sigma_\omega^{-1}F + \Sigma_\omega^{-1} \qquad 0 < n < N\\
&\nabla_{x_N}\nabla_{x_N}\Phi_R = H^T\Sigma_\eta^{-1}H + \Sigma_\omega^{-1}
\end{aligned}
$$

并且 Ω 正定， 否则存在 $x \neq 0$ 使得 $x^T\Omega x = 0$， 即

$$
0 = \frac{1}{2}\sum_{n=0}^{N-1}\|\Sigma_\eta^{-1/2}Hx_{n+1}\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\|\Sigma_\omega^{-1/2}(x_{n+1} - Fx_n)\|^2 + \frac{1}{2}\|C_0^{-1/2}x_0)\|^2
$$

因此 $x_0 = 0$ 并且 $x_{n+1} = Fx_n = 0$， 导出矛盾。

4. 证明

因为 Ω 正定，Ω_n 正定。

$$
m_N^{\text{smoothing}} = \int \rho(X_N|Y_N)x_Ndx_1dx_2\cdots dx_N = \int \rho(x_N|Y_N)x_Ndx_N = m_N^{\text{filtering}}
$$