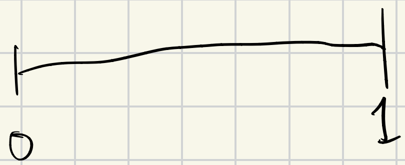
含参数的偏微分方程

$u(t,x)$ 偏微分方程的解

粗体 $\mathbf{u}(t) = \begin{bmatrix} u(t,x_1) \\ u(t,x_2) \\ \vdots \\ u(t,x_N) \end{bmatrix} \in \mathbb{R}^N$ 是空间离散后的向量

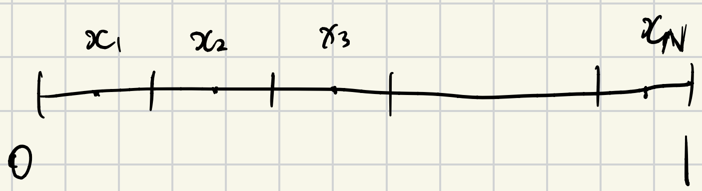
① 对流扩散方程

$$\partial_t u + \partial_x(au) = \partial_x(\nu\,\partial_x u)$$



$$\partial_t \int_0^1 u\,dx + \underbrace{\int_0^1 \partial_x(au)}_{0} = \underbrace{\int_0^1 \partial_x \nu\,\partial_x u}_{0}$$

周期性



$$\mathbf{U}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{bmatrix} \qquad u_i(t) = \frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(t,x)\,dx$$

$$\simeq u(t, x_i) + O(\Delta x_i^2)$$

守恒律：

$$\int_0^1 u\,dx = \sum \Delta x_i\, u_i(t)$$

$u_i(t)$ 如何随时间演化

$$\frac{d}{dt}u_i(t) = \frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial u(t,x)}{\partial t}\,dx$$

$$= \frac{1}{\Delta x_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \partial_x(\nu\partial_x u) - \partial_x(au)\,dx$$

$$= \frac{[\,(\nu\partial_x u) - (au)\,](x_{i+\frac{1}{2}}) - [\quad](x_{i-\frac{1}{2}})}{\Delta x_i}$$

$$:= \frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\Delta x_i}$$

$$\Delta x_i \frac{d u_i(t)}{dt} = \sum f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} = 0$$

$$\dot{u}_i = \left[\frac{u_i + u_{i+1}}{2} - \frac{u_i + u_{i-1}}{2}\right] / \Delta x$$

$$\dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

$$u_i^{n+1} = u_i^n + (u_{i+1}^n - u_{i-1}^n)\frac{\Delta t}{2\Delta x}$$

$$u_i^n \sim \varepsilon_i^n = e^{k x_i I} \quad (误差)$$

$$\varepsilon_i^{n+1} = \varepsilon_i^n + (\varepsilon_{i+1}^n - \varepsilon_{i-1}^n)\lambda$$

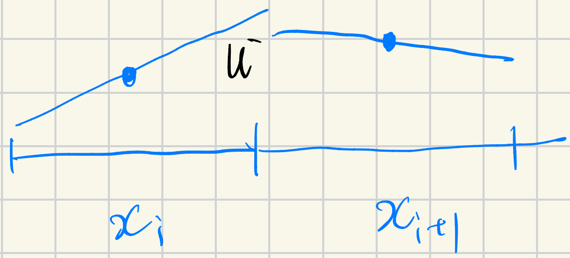
$$= \varepsilon_i^n\left(1 + (e^{k\Delta x I} - e^{-k\Delta x I})\lambda\right)$$

$$= \varepsilon_i^n\left(1 + 2\lambda \sin(\Delta x k)\, i\right)$$

$$|\varepsilon_i^{n+1}|^2 \geq |\varepsilon_i^n|^2\left(1 + [2\lambda \sin(\Delta x k)]^2\right)$$

限制器（重构）

$u_{i-1}(t)$ $u_i(t)$ $u_{i+1}(t)$ $\longrightarrow$ $u(t, x)$



$$u^- = u_i + \frac{\Delta x_i}{2} s_i \qquad s_i = \min\{\partial u\}$$

$$u^+ = u_{i+1} - \frac{\Delta x_{i+1}}{2} s_{i+1} \qquad s_{i+1} = \min\{\partial u\}$$

时间离散：

$$\frac{du}{dt} = f(u, t, x)$$

$$u(t+\Delta t) = u(t) + \int_t^{t+\Delta t} f(u, t, x)\, dt$$

向前欧拉方法　　截断误差 $O(\Delta t^2)N$

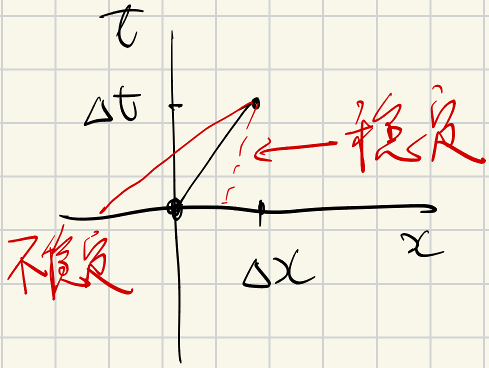
$$u(t+\Delta t) = u(t) + \Delta t\, f(u(t), t, x)$$

Runge Kutta 方法　　$N \sim \frac{1}{\Delta t}$

截断误差 $O(\Delta t^3)N$

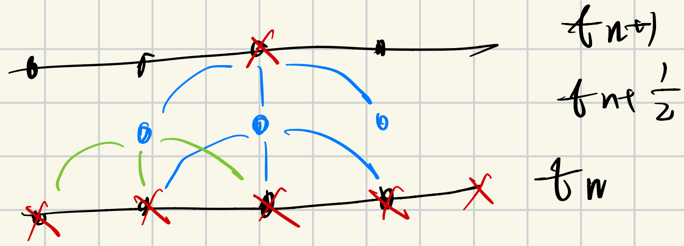
时间离散 CFL 条件：

信息传递

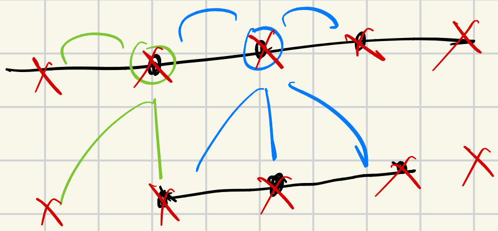


$$a \cdot \Delta t \le \Delta x$$

2nd Runge-Kutta 格式



隐式格式



扩散项

$$\dot{u}_i = \nu \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

$$u_i^{n+1} = u_i^n + \nu \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

$$\varepsilon_i^n \sim e^{k x_i I}$$

$$\varepsilon_i^{n+1} = \varepsilon_i^n\left(1 + \nu \frac{\Delta t}{\Delta x^2}\left(e^{k\Delta x I} - 2 + e^{-k\Delta x I}\right)\right)$$

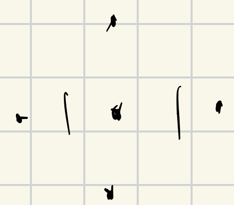
$$= \varepsilon_i^n\left((1-2\lambda) + 2\lambda\cos(k\Delta x)\right)$$

$$4\lambda \le 2 \quad \left(\lambda \le \frac{1}{2}\right)$$

Darcy 方程（有限差分方法）

$$\nabla \cdot (a \nabla u) = f$$

$$\partial_x (a\, \partial_x u)(x_i\ y_j)$$



$$a_{i-\frac{1}{2}j}\ \frac{u_{ij} - u_{i-1j}}{\Delta x} \qquad a_{i+\frac{1}{2}j}\ \frac{u_{i+1j} - u_{ij}}{\Delta x}$$

$$\begin{bmatrix} \\ \\ & A & \\ \\ \\ \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ \vdots \\ u_{21} \\ u_{22} \\ u_{23} \\ \vdots \\ u_{n1} \\ u_{n2} \\ \vdots \\ u_{nn} \end{bmatrix}$$

Darcy方程（有限元方法）

二阶可微 ⇒ 1阶可微（数学性质）

$$\int \partial_x(a\partial_x u) + \partial_y(a\partial_y u)\ \phi\ d\Omega$$

$$= \int \partial_x(a\partial_x u\phi) + \partial_y(a\partial_y u\phi) - a\partial_x u\,\partial_x\phi - a\partial_y u\,\partial_y\phi$$

$$= -\int a\partial_x u\,\partial_x\phi + a\partial_y u\,\partial_y\phi\ d\Omega \quad (\phi = 0 \text{ on } \partial\Omega)$$

$$= \int f\,\phi\ d\Omega$$

$$u = \sum_{i=1}^{N} u_i\,\phi_i(x, y)$$

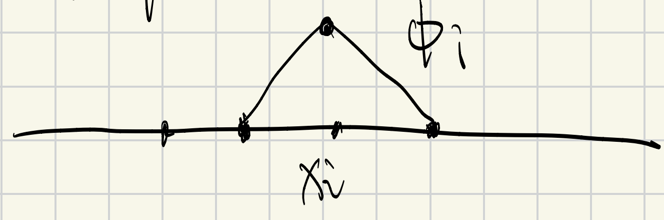
N个未知量

N个方程

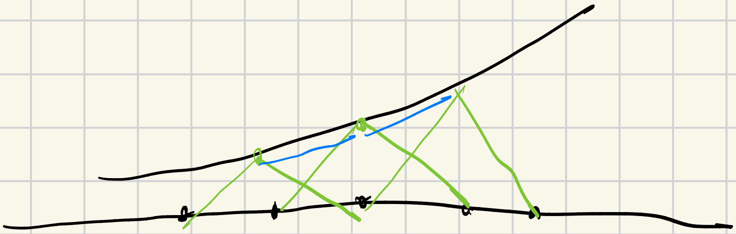
初期 ⇒ $\phi_i = \sin(ax+by)$ ......

ϕ_i 较为局部的函数

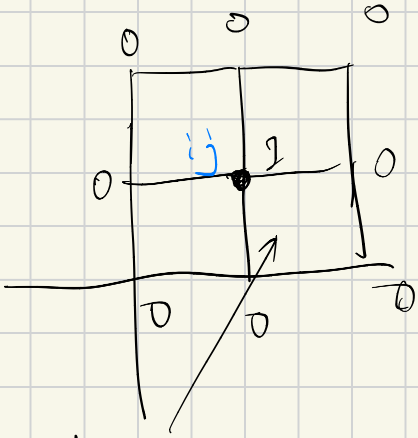
一维



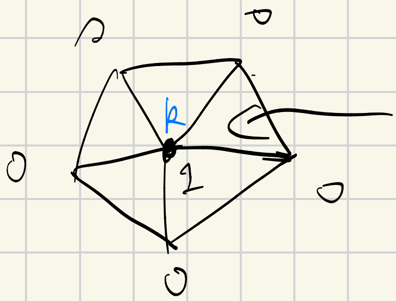
分片线性



二维



$$\phi_{ij}|_e = axy + bx + cy + d$$



$$\phi_k|_e = ax + by + c$$