PKU MODEL THEORY: LECTURE 1

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1. Propositional Logic

1.1. **Propositional language.**

Definition 1.1. A propositional logic is build from two components.

(1) Atomic Propositions: $A_1, A_2, ...$
(2) Logical symbols:
  (a) Logical connectives: '$\neg$' '$\wedge$', '$\vee$', '$\rightarrow$'.
  (b) Parenthesis: ')' and '('.

Definition 1.2. A language $\mathcal{L}$ in propositional logic is a set of atomic propositions, e.g. $\mathcal{L} = \{A_i : i \in \mathbb{N}\}$. We sometimes call $\mathcal{L}$ a **PL-language**.

Remark 1.3. One should think of an atomic proposition as a sentence;

(1) C_1 = "It is cloudy in Beijing".
(2) C_2 = "It is raining in Beijing".
(3) C_3 = "It is cold in Beijing".
(4) C_4 = "It is hot in Beijing".

Definition 1.4. Let $\mathcal{L}$ be a PL-language. Then an $\mathcal{L}$-model is a subset of $\mathcal{L}$, e.g. if $\mathcal{L} = \{C_1, C_2, C_3, C_4\}$, then $M = \{C_1, C_3\}$ is a model. Notice that M can be empty.

Question 1.5. How do we construction sentences in this logic?

Definition 1.6. The $\mathcal{L}$-sentences are defined as follows:

(1) Every elements of $\mathcal{L}$ is an $\mathcal{L}$-sentence. In other words, every atomic proposition is a sentence.
(2) If φ is a sentence, then $(\neg\varphi)$ is a sentence.
(3) If φ and ψ are sentences, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, and $(\varphi \rightarrow \psi)$ are sentences.
(4) A finite sequence of symbols is a sentence only if it can be shown to be a sentence by a finite number of applications of (1) – (3).

Example 1.7. Let $\mathcal{L} = \{A_1, A_2, A_3, A_4\}$.

(1) A_1 is a sentence.
(2) $(A_1 \wedge A_1)$ is a sentence.
(3) $(\neg(A_1 \wedge (A_2 \vee A_2)))$ is a sentence.
(4) $\neg A_1)$ is not a sentence.
(5) (((((((is not a sentence.
(6) $A_1 A_1 \wedge$ is not a sentence.

The definition of a sentence is importantly recursive. This allows us to apply structural induction.

Proposition 1.8. *Let $\mathcal{L}$ be a PL-language. Then for any $\mathcal{L}$-sentence φ, if we let*

$$L(\varphi) = \#\text{ of times the symbol `(' occurs in } \varphi.$$

and,

$$R(\varphi) = \#\text{ of times the symbol `)' occurs in } \varphi.$$

Then $L(\varphi) = R(\varphi)$.

Proof. This proof is via structural induction.

Base Case: Suppose that $\varphi = A$. Then $L(\varphi) = 0$ and $R(\varphi) = 0$ and so $L(\varphi) = R(\varphi)$.

Induction Hypothesis: Suppose that $L(\psi) = R(\psi)$ and $L(\theta) = R(\theta)$. WLOG, suppose that $L(\psi) = n$ and $L(\theta) = m$.

Negation: If $\varphi = (\neg\psi)$, then $L(\varphi) = L((\neg\psi)) = 1 + L(\psi) = 1 + n$. Also, $R(\varphi) = R((\neg\psi)) = 1 + R(\psi) = 1 + n$. Hence $L(\varphi) = R(\varphi)$.

Binary connectives: Let $\otimes \in \{\wedge, \vee, \rightarrow\}$. Suppose that $\varphi = (\psi \otimes \theta)$. Then $L(\varphi) = L((\psi \otimes \theta)) = 1 + n + m$. Moreover, $R(\varphi) = R((\psi \otimes \theta)) = n + m + 1$. Hence $L(\varphi) = R(\varphi)$.

By structural induction, we conclude that for any $\mathcal{L}$-sentence, $R(\varphi) = L(\varphi)$. □

Corollary 1.9. $\neg A_1)$ *is not a sentence.*

1.2. **Propositional Models.** Models concern whether or not a sentence is true *relative to that particular model*. What does it mean for a sentence φ to be true in M?

Definition 1.10. Let $\mathcal{L}$ be a PL-language, M an $\mathcal{L}$-models and φ a $\mathcal{L}$-sentence. The relation $M \models \varphi$ is read 'φ is true in M' and is defined as follows:

1. If $\varphi = A$, then $M \models A$ if and only if $A \in M$.
2. If $\varphi = (\neg\psi)$, then $M \models \varphi$ if and only if it is not the case that $M \models \psi$, or in other words $M \not\models \psi$.
3. If $\varphi = (\psi \wedge \theta)$, then $M \models \varphi$ if and only if $M \models \psi$ and $M \models \theta$.
4. If $\varphi = (\psi \vee \theta)$, then $M \models \varphi$ if and only if $M \models \psi$ or (inclusively) $M \models \theta$.
5. If $\varphi = (\psi \rightarrow \theta)$, then $M \models \varphi$ if and only if either $M \models \neg\psi$ or $M \models \psi \wedge \theta$.

We now discuss truth assignments and valuations:

Definition 1.11. $I(\varphi)$ is the collection of atomic propositions which occur in φ.

Definition 1.12. Let $\mathcal{L}$ be a PL-language, $B \subseteq \mathcal{L}$, and $g : B \rightarrow \{T, F\}$. We call g a *truth assignment* and we define the map $v_g : \{\varphi \text{ is an } \mathcal{L}\text{-sentence } | I(\varphi) \subseteq B\} \rightarrow \{T, F\}$ recursively as follows: Suppose that $I(\varphi) \subseteq B$,

1. If $\varphi = A$, then $v_g(\varphi) = g(A)$.
2. If $\varphi = (\neg\psi)$, then $v_g(\varphi) = T$ if $v_g(\psi) = F$ and $v_g(\varphi) = F$ if $v_g(\psi) = T$.
3. If $\varphi = (\psi \wedge \theta)$, then $v_g(\varphi) = T$ if $v_g(\psi) = v_g(\theta) = T$. Otherwise, $v_g(\varphi) = F$.
4. If $\varphi = (\psi \vee \theta)$, then $v_g(\varphi) = T$ if $v_g(\psi) = T$ or (inclusively) $v_g(\theta) = T$. Otherwise $v_g(\varphi) = F$.
5. If $\varphi = (\psi \rightarrow \theta)$, then $v_g(\varphi) = T$ if $v_g(\psi) = F$ or $v_g(\psi) = v_g(\theta) = T$. Otherwise, $v_g(\varphi) = F$.

Definition 1.13. Let φ be an $\mathcal{L}$-sentence.

1. φ is said to be valid if φ is true in every $\mathcal{L}$-model.
2. φ is said to be satisfiable if there exists a $\mathcal{L}$-model M such that $M \models \varphi$.

(3) φ is said to be not satisfiable if there does not exist an $\mathcal{L}$-model M such that $M \models \varphi$.

Lemma 1.14. *Let φ be an $\mathcal{L}$-sentence. Suppose that $I(\varphi) \subseteq B \subseteq \mathcal{L}$ and $f : B \to \{T, F\}$ is a truth assignment. Then $v_f(\varphi) = T$ if and only if $M_f \models \varphi$ where $M_f = \{A \in \mathcal{L} : f(A) = T\}$.*

Proof. By structural induction. □

Lemma 1.15. *Let φ be an $\mathcal{L}$-sentence such that $I(\varphi) \subseteq B \subseteq \mathcal{L}$ and $g : B \to \{T, F\}$ is a truth assignment. Then $v_g(\varphi) = v_h(\varphi)$ where $h = g|_{I(\varphi)}$.*

Proof. By structural induction. □

Theorem 1.16. *Let φ be a $\mathcal{L}$-sentence. Then the following are equivalent.*

(1) φ is valid.
(2) For any truth assignment $f : I(\varphi) \to \{T, F\}$, $v_f(\varphi) = T$.

Proof. (1) $\implies$ (2). Suppose that φ is valid. Then for any $\mathcal{L}$-model M, $M \models \varphi$. In particular, $M_f \models \varphi$ for any $f : I(\varphi) \to \{T, F\}$. By Lemma 1.14, we conclude that $v_f(\varphi) = T$ for any $f : I(\varphi) \to \{T, F\}$.

(2) $\implies$ (1). Let M be any $\mathcal{L}$-model. Consider the truth assignment $g_M : \mathcal{L} \to \{T, F\}$ where $g_M(A) = T$ if and only if $A \in M$. Consider the truth assignment $h : I(\varphi) \to \{T, F\}$ where $h = g_M|_{I(\varphi)}$. By assumption, $v_h(\varphi) = T$. By Lemma 1.15, $g_M(\varphi) = T$. By Lemma 1.14, $M_{g_M} \models \varphi$. However, we notice that $M_{g_M} = M$ since $A \in M_{g_M}$ if and only if $g_M(A) = T$ if and only if $A \in M$. Therefore, $M \models \varphi$. □

Hence, to check if a sentence φ is valid, one only needs to check a finite amount of data. One can arrange this data in a **truth table**.

2. Formal Proofs

Exercise 2.1. *If $M \models (\psi \to \theta)$ and $M \models \psi$, then $M \models \theta$.*

Definition 2.2. A collection of $\mathcal{L}$-sentences is called a $\mathcal{L}$-theory or just a theory when $\mathcal{L}$ is unambiguous.

Definition 2.3. Let $\mathcal{L}$ be a language and Σ an $\mathcal{L}$-theory. Let φ be an $\mathcal{L}$-sentence. We write '$\Sigma \vdash \varphi$' and say Σ *proves* φ if there exists a finite sequence $\theta_1, ..., \theta_n$ of $\mathcal{L}$-sentences such that

(1) $\theta_n = \varphi$.
(2) For each $m \leq n$, either
  (a) θ_m is valid.
  (b) θ_m is in Σ.
  (c) θ_m is inferred from two previous sentences in the sequence, i.e. there exists $k, l < m$ such that $\theta_k = (\psi \to \theta_m)$ and $\theta_l = \psi$. This inference is called *modus ponnes*.

The sequence $\theta_1, ..., \theta_n$ is called a *proof* or a *deduction* from Σ to φ.

Definition 2.4. We say that φ is a tautology if $\emptyset \vdash \varphi$ and usually just write $\vdash \varphi$.

Proposition 2.5. *An $\mathcal{L}$-sentence φ is valid if and only if φ is a tautology.*

Proof. ($\Rightarrow$) Suppose that φ is valid. Consider the sequence θ_1 where $\theta_1 = \varphi$. This is a deduction of φ from $\emptyset$. So $\vdash \varphi$.

($\Leftarrow$) Suppose that φ is a tautology. Let $\theta_1, ..., \theta_n$ be a proof of φ from $\emptyset$. We prove that φ is valid via induction on the length of the proof.

Suppose that $n = 1$. Then θ_1 is a proof of φ. So $\theta_1 = \theta_n = \varphi$. By construction of formal proofs, and since $\Sigma = \emptyset$, we conclude that φ must be valid.

Suppose that if ψ has a proof of length less than or equal to n (from $\emptyset$), then ψ is valid. Let $\theta_1, ..., \theta_{n+1}$ be a proof of φ.

(1) If θ_{n+1} is valid, done.
(2) If θ_{n+1} is inferred via modus ponens, then there exists $k, l < n+1$ such that $\theta_k = (\psi \to \theta_{n+1})$ and $\theta_l = \psi$. Notice that $\theta_1, ..., \theta_k$ is a proof of θ_k (from $\emptyset$) and $\theta_1, ..., \theta_l$ is a proof of θ_l (from $\emptyset$). By our induction hypothesis, we conclude that θ_l and θ_k are valid. Let M be any $\mathcal{L}$-model. Then $M \models \theta_k$ and $M \models \theta_l$. Hence $M \models (\psi \to \theta_{n+1})$ and $M \models \psi$. By the exercise above, we conclude that $M \models \theta_{n+1}$. Since M was arbitrary, θ_{n+1} is valid. $\square$