PKU MODEL THEORY NOTES

KYLE GANNON

1. Ax-Grothendieck

1.1. Fields.
Let E be a field extension of F.

Definition 1.1. Let $S \subseteq E$. We say that S is algebraically independent over F is for all non-zero polynomials $p(x_1, ..., x_n) \in F[x_1, ..., x_n]$ and $s_1, ..., s_n \in S$ (all distinct) we have that $p(s_1, ..., s_n) \neq 0$.

Definition 1.2. We say that a subset S of E is a transcendence basis if S is algebraically independent over the prime field and S is maximal, i.e. for any extension $S' \supseteq S$, S' is algebraically dependent.

Example 1.3. If we consider $\bar{\mathbb{Q}}(\pi)$, then $S = \{\pi\}$ is algebraically independent.

Fact 1.4. *If S and S' are transcendence basis for E, then $|S| = |S'|$.*

Fact 1.5. *If E is an algebraically closed field of size κ, then E has a transcendence basis of size κ.*

Fact 1.6 (Steinitz)**.** *Algebraically closed fields are determined up to isomorphism by their characteristic and the size of their transcendence basis.*

1.2. Model theory.
We recall that theory of algebraically closed fields, **ACF**.

Definition 1.7. **ACF** is a theory in the language of rings $\mathcal{L}_{ring} = \{+, \times, 0, 1\}$ with the following axioms:

(1) $\forall x \forall y (x + y = y + x)$.
(2) $\forall x \forall y \forall z ((x + y) + z = (x + (y + z)))$.
(3) $\forall x \exists y (x + y = y + x = 0)$.
(4) $\forall x (x + 0 = 0 + x = x)$.
(5) $\forall x \forall y (x \times y = y \times x)$.
(6) $\forall x \forall y \forall z ((x \times y) \times z = (x \times (y \times z)))$.
(7) $\forall x \exists y (x \neq 0 \rightarrow (x \times y = y \times x = 1))$.
(8) $\forall x (x \times 1 = 1 \times x = x)$.
(9) $\forall x \forall y \forall z (x \times (y + z) = x \times y + x \times z)$.
(10) $0 \neq 1$.

For any prime p, we let $\mathbf{ACF}_p = \mathbf{ACF} \cup \{\underbrace{1 + ... + 1}_{p-times} = 0\}$. We also define the theory $\mathbf{ACF}_0 = \mathbf{ACF} \cup \{\underbrace{1 + ... + 1}_{p-times} \neq 0 : p \text{ is prime}\}$.

Proposition 1.8. *For any $p \in \mathbb{P}$ or $p = 0$, we have that the theory $\mathbf{ACF}_p$ is complete.*

Proof. We claim that $\mathbf{ACF}_p$ is $\aleph_1$-categorical. Let N_1 and N_2 be two models of $\mathbf{ACF}_p$ of size $\aleph_1$. Then they both have a transcendence basis of size $\aleph_1$. By Fact 1.6, they are isomorphic.

One can check that these theories have no finite models [Hint: consider $\forall x \exists y(y^2 = x) \wedge (1+1 \neq 0)$ or some variant]. Hence by Vaught's test, we conclude that $\mathbf{ACF}_p$ is complete. □

Lemma 1.9 (Lefschetz Principle)**.** *Let φ be a setnence in the language of rings, $\mathcal{L}_{rings}$. Then the following are equivalent:*

(1) For arbitrarily large primes p, $\mathbf{ACF}_p \models \varphi$.
(2) $\mathbf{ACF}_0 \models \varphi$.
(3) $\mathbb{C} \models \varphi$.

Proof. We notice that (2) $\equiv$ (3) by completeness of $\mathbf{ACF}_0$ and the fundamental theorem of algebra, i.e. $\mathcal{C} \models \mathbf{ACF}_0$. (1) $\equiv$ (2) is a direct application of the compactness theorem. □

Definition 1.10. Fix a language $\mathcal{L}$. We say that a sentence is a Π_n-sentence if it is of the form

$$\underbrace{\forall \bar{x}_1, \exists \bar{x}_2, ...}_{(n-1)-alternations} \quad \psi(\bar{x}_1, ..., \bar{x}_n),$$

where $\psi(\bar{x}_1, ..., \bar{x}_n)$ is quantifier free. We say that a sentence φ is a Σ_n-sentence if it is of the form

$$\underbrace{\exists \bar{x}_1, \forall \bar{x}_2, ...}_{(n-1)-alternations} \quad \psi(\bar{x}_1, ..., \bar{x}_n),$$

where $\psi(\bar{x}_1, ..., \bar{x}_n)$ is quantifier free. For example, if $\psi(x_1, x_2, x_3)$ is quantifier free, then

$$\forall x_1 \forall x_3 \exists x_2 \psi(x_1, x_2, x_3),$$

is a Π_2-sentence.

Lemma 1.11. *Suppose that $M = \bigcup_{i \in I} M_i$ such that*

(1) For each i, M_i is a substructure of M.
(2) For each i, M_i is a substructure of M_{i+1}.

For any Π_2 sentence φ, if $M_i \models \varphi$ for every $i < \omega$, then $M \models \varphi$.

Proof. Suppose that

$$\varphi = \forall x_1, ..., x_n \exists y_1, ..., y_m \psi(\bar{x}, \bar{y}),$$

where $\psi(\bar{x}, \bar{y})$ is quantifier free. Let $a_1, ..., a_n \in M$. There exists some $\beta < \omega$ such that $a_1, ..., a_n \in M_\beta$. By our hypothesis, $M_\beta \models \psi$. Hence,

$$M_\beta \models \exists y_1, ..., y_m \psi(a_1, ..., a, y_1, ..., y_m).$$

By definition of satisfaction, this implies

$$M_\beta \models \psi(a_1, ..., a_n, b_1, ..., b_m),$$

for some $b_1, ..., b_m \in M_\beta$. Since $\psi(\bar{x}, \bar{y})$ is quantifier free, we have that $M \models \varphi(\bar{a}, \bar{b})$. Therefore we have shown that $M \models \forall \bar{x} \exists \bar{y} \psi(\bar{x}, \bar{y})$. □

Example 1.12. We can write $(\mathbb{N}, \leq) = \bigcup_{n > 4}(\{1, ..., 4\}, \leq)$. Notice that Σ_2 sentences do not necessarily work.

1.3. **Finite fields.** For every prime number p and every natural numbers $n \geq 1$, there exists a unique field (up to isomorphism) of cardinality $|p^n|$. This field has characteristic p and we denote it F_{p^n}. Hence the collection of all finite fields of characteristic p are of the form $\{F_p, F_{p^2}, F_{p^3}, ...\}$. Moreover, there is an injective homomorphism from a field F_{p^n} to F_{p^m} if and only if $n|m$.

Remark 1.13. There are two ways one can construct an algebraically closed fields of characteristic p. One is to use the *direct limit* construction. For every pair of $n \geq m \geq 1$ where $m|n$, choose a maps $g_{m,n} : F_{p^m} \to F_{p^n}$ such that

(1) $g_{m,m} = id_{F_{p^m}}$.
(2) $g_{m,n}$ is an injective homomorphism.
(3) for any k such that $n < k$ and $n|k$, we have that $g_{m,k} = g_{n,k} \circ g_{m,n}$.

Then the direct limit is the set

$$\lim_{\rightarrow} F_{p^n} = \bigsqcup_{n \in \mathbb{N}} F_{p^n} / \sim$$

where if $a \in F_{p^n}$ and $b \in F_{p^m}$ then $a \sim b$ if and only if there exists some k such that $g_{n,k}(a) = g_{m,k}(b)$. In other words, they are eventually equal. Choosing this set of maps so that everything is consistent is a little bit of a pain. It turns out that this is a model of $\mathbf{ACF}_p$.

Remark 1.14. The second way is the same as above, but we consider a smaller collection of structures. In particular, consider the set $\{F_{p^{n!}} : n \geq 1\}$. Thence we have a sequence of homomorphisms,

$$F_p \to F_{p^{2!}} \to ... \to F_{p^{n!}} \to ...$$

Since these are injective homomorphisms, we can think of these structures nested inside on another, i.e.

$$F_p \subseteq F_{p^{2!}} \subseteq ... \subseteq F_{p^{n!}} \subseteq ...$$

Then the direct limit is simply the union. Furthermore, one can check that this is also a model of $\mathbf{ACF}_p$. We let $\bar{\mathbb{F}}_p = \bigcup_{n \geq 1} F_{p^{n!}}$

1.4. **Ax-Grothendieck.**

Lemma 1.15. *Let φ be a Π_2 sentence in the language of rings and suppose that φ is true in all finite fields. Then $\mathbf{ACF}_p \models \varphi$ for every prime p. Moreover, we have that $\mathbf{ACF}_0 \models \varphi$ and thus $\mathbb{C} \models \varphi$.*

Proof. By Remark 1.14, we can construct a model of $\mathbf{ACF}_p$ as an increasing union of finite fields. Since φ is true in all finite fields, it is true in all the models in our construction. Since φ is Π_2, we may apply Lemma 1.11 and conclude that $\bar{\mathbb{F}}_p \models \varphi$. By completeness, $\mathbf{ACF}_p \models \varphi$. Since p was arbitrary, we know that $\mathbf{ACF}_p \models \varphi$ for arbitrarily large p. By the Lefschetz Principle, we are done. □

Fact 1.16 (Quantifiers)**.** *Suppose that φ and ψ are formulas. Suppose furthermore that x does not appear freely in ψ. Then*

(1) $(\forall x \varphi) \to \psi$ is equivalent to $\exists x(\varphi \to \psi)$
(2) $(\exists x \varphi) \to \psi$ is equivalent to $\forall x(\varphi \to \psi)$

Moreover, if y does not appear freely in φ, then

(1) $\varphi \to (\exists y \psi)$ is equivalent to $\exists x(\varphi \to \psi)$
(2) $\varphi \to (\forall y \psi)$ is equivalent to $\forall x(\varphi \to \psi)$

Theorem 1.17 (Ax-Grothendieck)**.** *Suppose that $P : \mathbb{C}^m \to \mathbb{C}^m$ is a polynomial map, i.e.*

$$P(x_1, ..., x_m) = (P_1(x_1, ..., x_m), ..., P_m(x_1, ..., x_m)),$$

If P is injective, then P is surjective.

Proof. We construct a Π_2-sentence φ such that

(1) φ is true in every finite field.
(2) If φ is true, then every polynomial map (of a certain form) which is injective is also surjective.
(3) By the previous lemma, we will conclude that the statement holds in $\mathbb{C}$.

We write the proof in the case where $m = 1$. The proof is the same for higher dimensions. Suppose that the degree of P is n, so $P(x) = a_n x^n + ... + a_1 x + a_0$. Then we want to write, "$P(x)$ is injective implies $P(x)$ is surjective". So this is the same as,

$$(\forall x \forall y (P(x) = P(y) \to x = y) \to \forall z \exists w (P(w) = z))$$

Formally, we say, "For any polynomial of degree n, if the map is injecitve, then it is surjective",

$$\begin{aligned} \forall v_0, ..., v_n (\forall x \forall y (v_n x^n + ... + v_1 x + v_0 = v_n y^n + ... + v_1 y + v_0 \to x = y) \\ \to \forall z \exists w (v_n w^n + ... v_1 w + v_0 = z)) \end{aligned}$$

Moving the quantifier out (via the previous fact), we see that this is equivalent to:

$$\begin{aligned} \forall v_0, ..., v_n \forall z \exists w \exists x \exists y ((v_n x^n + ... + v_1 x + v_0 = v_n y^n + ... + v_1 y + v_0 \to x = y) \\ \to (v_n w^n + ... v_1 w + v_0 = z)). \end{aligned}$$

Let the sentence above be φ. Notice that

(1) φ is a Π_2 sentence.
(2) φ is true in any finite field. Indeed, φ says that any polynomial map (of degree n) from a finite set to a finite set which is injective is also surjective. This is true because any function from a finite set to a finite set which is injective is also surjective.
(3) By the preivous lemma, we see that $\mathbb{C} \models \varphi$.

□