PKU MODEL THEORY NOTES

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1. Stone spaces & prime models

Fact 1.1. *Let M be an $\mathcal{L}$-structure and $M \models \forall \bar{x}\theta(\bar{x})$. Then for every $p \in S_{\bar{x}}(M)$, $\theta(\bar{x}) \in p$.*

Definition 1.2. Fix an $\mathcal{L}$-structure M, a tuple $\bar{x} = x_1, ..., x_n$ and a set $A \subseteq M$. Consider $S_{\bar{x}}(A)$. This space is naturally a topological space. For every formula $\theta(\bar{x}) \in \mathcal{L}_{\bar{x}}(A)$, we let

$$[\theta(\bar{x})] = \{p \in S_{\bar{x}}(A) : \theta(\bar{x}) \in p\}.$$

Then our collection of basic open subsets of $S_{\bar{x}}(A)$ is precisely $\{[\theta(\bar{x})] : \theta(\bar{x}) \in \mathcal{L}_x(A)\}$.

Definition 1.3. Suppose that X is a topological space. We say that X is totally disconnected if for every $a, b \in X$, there exists a clopen set C such that $a \in C$ and $b \in X\backslash C$. We recall that a set is called clopen if it is both "closed and open".

Proposition 1.4. *For fixed M, $\bar{x} = x_1, ..., x_n$ and $A \subseteq M$, the space $S_{\bar{x}}(A)$ is a compact Hausdorff totally disconnected space.*

Proof. We first show that the space is both Hausdorff and totally disconnected. Fix distinct $p, q \in S_{\bar{x}}(A)$. Since $p \neq q$, there exists a formula $\theta(\bar{x}) \in \mathcal{L}_{\bar{x}}(A)$ such that $\theta(\bar{x}) \in p$ and $\theta(\bar{x}) \notin q$. Since q is a complete type, $\neg\theta(\bar{x}) \in q$. By definition, this implies $p \in [\theta(\bar{x})]$ and $q \in [\neg\theta(\bar{x})]$. Hence the space is both Hausdorff and totally disconnected.

We now show that the space is compact. Suppose that $U = \bigcup_{i \in I}[\theta_i(\bar{x})]$ is an open cover of $S_{\bar{x}}(A)$. This implies that $\bigcap_{i \in I}[\neg\theta_i(\bar{x})] = \emptyset$. This implies that

$$Th_{\mathcal{L}_A}(M) \cup \{\neg\theta_i(\bar{d}) : i \in I\}$$

is inconsistent where $Th_{\mathcal{L}_A}(M)$ is the theory of M in the language $\mathcal{L} \cup \{c_a : a \in A\}$ and $\bar{d} = d_1, ..., d_n$ are new constant symbols. Since proofs are finitary, we conclude that

$$Th_{\mathcal{L}_A}(M) \cup \{\neg\theta_j(\bar{d}) : j = 1, ..., n\},$$

for some finite subset $J \subset I$. This implies that

$$Th_{\mathcal{L}_A}(M) \vdash \neg \bigwedge_{j=1}^{n} \neg\theta_j(\bar{d}),$$

and so,

$$Th_{\mathcal{L}_A}(M) \vdash \forall x \bigvee_{j=1}^{n} \theta_j(\bar{x}).$$

We claim that this implies that $\{[\theta_j(\bar{x})] : j \in J\}$ is a finite subcover of U. $\square$

Definition 1.5. If $\bar{a} \in M^n$, we let

$$\text{tp}(\bar{a}/A) = \{\varphi(\bar{x}) : \varphi(\bar{x}) \in \mathcal{L}_{\bar{x}}(A) \text{ and } M \models \varphi(\bar{a})\}.$$

We remark that $\text{tp}(\bar{a}/A)$ is a complete type in $S_{\bar{x}}(A)$ where $\bar{x} = x_1, ..., x_n$.

Definition 1.6. Let X be an arbitrary topological space and $Y \subseteq X$. We say that Y is dense inside X if for every non-empty open subsets O of X, $Y \cap O \neq \emptyset$.

The next proposition tells us essentially that the model M is dense inside the type space $S_x(M)$

Proposition 1.7. *$\{\text{tp}(\bar{a}/M) : \bar{a} \in M^n\}$ is a dense subset of $S_{\bar{x}}(M)$ where $\bar{x} = x_1, ..., x_n$.*

Proof. It suffices to prove this statement for basic opens. So, fix $[\theta(\bar{x})]$ a non-empty open subset of $S_{\bar{x}}(A)$. Since it is non-empty, there is some $p \in [\theta(\bar{x})]$. Hence $\theta(\bar{x}) \in p$. Since p is a type, by definition it is finitely satisfiable in M. In particular, there exists $\bar{d} \in M^n$ such that $M \models \theta(\bar{d})$. Hence $\theta(\bar{x}) \in \text{tp}(\bar{d}/M)$ and so $\text{tp}(\bar{d}/M) \in [\theta(\bar{x})]$. □

Fact 1.8. *Suppose that $A \subseteq B \subseteq M$. There the map $\pi_{B,A} : S_{\bar{x}}(B) \to S_{\bar{x}}(A)$ via $p \to p|_{\mathcal{L}_{\bar{x}}(A)}$ is both surjective and continuous.*

Fact 1.9. *Suppose that $A \subseteq M$ and $f : M \to N$ is an elementary embedding. This induceds a map $\tilde{f} : S_{\bar{x}}(A) \to S_{\bar{x}}(f(A))$ via*

$$\tilde{f}(p) = \{\varphi(x, f(\bar{d})) : \varphi(x, \bar{d}) \in p\}.$$

This map is a homeomorphism.

Definition 1.10. Fix $A \subseteq M$. We say that $p \in S_{\bar{x}}(A)$ is isolated if $\{p\}$ is an open subset of $S_{\bar{x}}(A)$.

Proposition 1.11. *Let $p \in S_{\bar{x}}(A)$. The following are equivalent:*

(1) p is isolated.
(2) $\{p\} = [\theta(\bar{x})]$ for some $\theta(\bar{x}) \in \mathcal{L}_{\bar{x}}(A)$.
(3) There exists some $\theta(\bar{x}) \in p$ such that for every $\psi(\bar{x}) \in p$,

$$Th_{\mathcal{L}_A}(M) \vdash \forall \bar{x}(\theta(\bar{x}) \to \psi(\bar{x})).$$

Proof. (1) $\iff$ (2) follows directly from definitions. Let's prove (2) implies (3). We assume that $\{p\} = [\theta(\bar{x})]$. We show actually that for any $\psi(\bar{x}) \in p$, the appropriate theory proves that $\forall \bar{x} \theta(\bar{x}) \to \psi(\bar{x})$. So fix $\psi(\bar{x}) \in p$ and suppose, towards a contradiction, that

$$Th_{\mathcal{L}_A}(M) \not\vdash \forall \bar{x}(\theta(\bar{x}) \to \psi(\bar{x})).$$

Then we have that

$$Th_{\mathcal{L}_A}(M) \cup \{\exists \bar{x}(\theta(\bar{x}) \to \psi(\bar{x}))\},$$

is consistent. Choose some elementary extension N such that $M \prec N$ and $N \models \exists \bar{x}(\theta(\bar{x}) \to \psi(\bar{x}))$. Then $N \models \theta(\bar{d}) \wedge \neg\psi(\bar{d})$ for some $\bar{d} \in N^n$. Consider $q = \text{tp}(\bar{d}/M)$. Then

(1) By construction, $\neg\psi(\bar{x}) \in q$.
(2) Since $\theta(\bar{x}) \in q$, we have that $q \in [\theta(\bar{x})]$. This implies that $p = q$ and so $\psi(x) \in q$.

Hence, we have a contradiction.

For (3) implies (2), we leave this as an exercise to the reader. □

Remark 1.12. If we are given two different models of a complete theory, then the types over the empty set are exactly the same. Lets be a little more precise: Fix T a complete first order theory.

(1) Say we fix a model M of T. Then we can consider $S_{\bar{x}}(\emptyset)$ which are the collection of types over the empty set. Recall that a type is just a maximally consistent collection of formulas which are finitely satisfiable in M. So, for now we will write this as $S^M_{\bar{x}}(\emptyset)$.
(2) If I have another model N of T, I can do the same process as above. In particular, I can consider the types over the empy set relative to this model, which we write as $S^N_{\bar{x}}(\emptyset)$.
(3) Check: $S^M_{\bar{x}}(\emptyset) = S^N_{\bar{x}}(\emptyset)$. Thus, types over the empyset are independent of a choice of model. Therefore, we will write this collection, $S^M_{\bar{x}}(\emptyset)$ simply as $S_{\bar{x}}(T)$.

Proposition 1.13. *Let T be a complete theory with infinite models. Suppose that $p \in S_{\bar{x}}(T)$ and p is isolated. Then for any model $M \models T$, there exists some $b \in M^n$ such that $b \models p$*

Proof. Since p is isolated, there exists a formula $\theta(\bar{x})$ such that for any $\psi(\bar{x}) \in p$, $T \vdash \forall\bar{x}(\theta(\bar{x}) \to \psi(x))$. We note that $T \vdash \exists\bar{x}(\theta(\bar{x}))$. Thus, if $N \models T$, then $N \models \exists\bar{x}(\theta(\bar{x}))$ and so there exists $\bar{d} \in N^n$ such that $N \models \theta(\bar{d})$. Hence $N \models \psi(\bar{d})$ for every $\psi(\bar{x}) \in p$. Thus, $\bar{d} \models p$. □

Theorem 1.14 (Omitting Types Theorem)**.** *Let $\mathcal{L}$ be a countable language, T be a complete $\mathcal{L}$-theory with infinite models, and $p \in S_{\bar{x}}(T)$ such that p is not isolated. Then there exists a countable model $M \models T$ such that the type p is omitted from M, i.e. p is not realized in M, i.e. for every $b \in M^n$, there exists a formula $\theta(\bar{x}) \in p$ such that $M \models \neg\theta(\bar{b})$.*

2. Prime models

Definition 2.1. We say that M is a prime model of a theory T if for any model N of T, there exists an elementary embedding $f : M \to M$.

Example 2.2. Here are some examples of theories with prime models:

(1) Consider the language of equality and T the theory which says, "I have infinitely many elements". Then any countable infinite set is a prime model of this theory. Indeed, any injective map in an elementary embedding.
(2) When T is the theory of algebraically closed fields of characteristic 0, then $\bar{\mathbb{Q}}$, the algebraic closure of the rationals, is a prime model of this theory. One can construct an injective ring homomorphism from $\bar{\mathcal{Q}}$ to an algebraically closed field. Quantifier elimination of $\mathbf{ACF}_0$ implies that this injective ring homomorphism is an elementary embedding.
(3) Consider true arithmetic, i.e. the theory of $\mathbb{N}$ in the language $\mathcal{L} = \{+, \times, 0, 1, <\}$. Then the standard model, i.e. $(\mathbb{N}, +, \times, 0, 1, <)$ with the usual interpretations, is a prime model. One can prove this using Tarski-Vaught and the fact that elements in $\mathbb{N}$ can be written as $1 + ... + 1$.

Remark 2.3. Let T be a complete theory with infinite models (in a countable language) and consider $p \in S_{\bar{x}}(T)$. If M is a prime model, and $a \models p$ then every model of T realizes p (why?). Hence the type p cannot be omitted. By the omitting types theorem, it follows that p is isolated.

Definition 2.4. We say that $M \models T$ is atomic if for every $\bar{a} \in M^n$, the type $\text{tp}(\bar{a}/\emptyset) \in S_{\bar{x}}(T)$ is isolated.

Remark 2.5. Prime models are atomic by Remark 2.3.

Proposition 2.6. *Let $\mathcal{L}$ be a countable language and T be a complete theory with infinite models. Then $M \models T$ is prime if and only if it is countable and atomic.*

Proof. We have already proved that prime implies atomic. We now show that atomic and countable implies prime. Let M be a countable atomic model of T and let N be another model. We construct a map $f : M \to N$. Let $m_1, m_2, ...$ be an enumeration of M. For each k, we let $\theta_k(x_1, ..., x_k)$ isolate the type $\text{tp}(m_1, ..., m_k/\emptyset) \in S_{\bar{x}}(T)$. We now build f.

(1) At step one, we consider $\{m_1\}$. We know that $\text{tp}(m_1/\emptyset)$ is isolated by $\theta_1(x_1)$. So $T \vdash \exists x_1 \theta_1(x_1)$ and so $N \models \exists x_1 \theta_1(x_1)$. Take $n_1 \in N$ such that $N \models \exists \theta_1(n_1)$ and let $f_1(m_1) = n_1$.
(2) At step $k+1$, suppose we have constructed f_k. The map $f_k : \{m_1, ..., m_k\} \to \{n_1, ..., n_k\}$ such that
$$M \models \theta_k(m_1, ..., m_k) \text{ and } N \models \theta_k(n_1, ..., n_k)$$
We now construct f_{k+1} with domain $\{m_1, ..., m_{k+1}\}$. By assumption, we have that $\text{tp}(m_1, ..., m_{k+1}/\emptyset)$ is isolated by $\theta_{k+1}(x_1, ..., x_{k+1})$. Hence, we have that $\exists y \theta(x_1, ..., x_k, y) \in \text{tp}(m_1, ..., m_k/\emptyset)$. By the equivalence of isolation, we see that
$$T \vdash \forall \bar{x}(\theta_k(\bar{x}) \to \exists y \theta_{k+1}(\bar{x}, y)).$$
Hence,
$$N \models \exists y \theta_{k+1}(n_1, ..., n_k, y),$$
and so we can choose n_{k+1} such that $N \models \theta_{k+1}(n_1, ..., n_k, n_{k+1})$. We let $f_{k+1}(m_{k+1}) = n_{k+1}$.

We let $f = \bigcup_{n<\omega} f_n$. By Tarski-Vaught, we claim that f is an elementary embedding. □