PKU MODEL THEORY NOTES

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This entire collection of notes comes almost directly from portions of Chapter 4 of Marker's Introduction to model theory.

Theorem 0.1. *Let $\mathcal{L}$ be a countable language and T a complete $\mathcal{L}$-theory with infinite models. Then the followsing are equivalent:*

1. *T has a prime model.*
2. *T has an atomic model.*
3. *The isolated types in $S_n(T)$ are dense for each $n \geq 1$.*

Proof. We proved $(i) \iff (ii)$ last lecture.

We first prove that $(ii) \implies (iii)$. Let M be an atomic model of T. Let O be a non-empty open subset of $S_n(T)$. Then

$$O = \bigcup_{i \in I} [\theta_i(\bar{x})].$$

Since O is non-empty, one of the $[\theta_i(\bar{x})]$ is non-empty. Hence $T \vdash \exists \bar{x}\theta(\bar{x})$. Then $M \models \exists \bar{x}\theta(\bar{x})$ and so for some $\bar{a} \in M^n$, $M \models \theta(\bar{a})$. Since M is atomic, the type $\text{tp}(\bar{a}/\emptyset)$ is isolated. Moreover, $\text{tp}(\bar{a}/\emptyset) \in [\theta_i(\bar{x})]$. This completes the proof.

$(iii) \implies (ii)$. This direction is by a careful Henkin construction. Suppose that the isolated types are dense. We will build an atomic model of T. Let C be a collection of countably many new constant symbols, $C = \{c_0, c_1, ...\}$ and $\mathcal{L}^* = \mathcal{L} \cup C$. Let $\phi_0, \phi_1, ...$ be an enumeration of all the $\mathcal{L}^*$-sentences. We build a sequence $\theta_0, \theta_1, ...$ of $\mathcal{L}^*$-sentences such that

1. $T^* = T \cup \{\theta_i : i = 0, 1, 2, ...\}$ is complete, satisfiable, and has witnesses in C.
2. The canonical model of T^*, namely $C/\sim$, is atomic.

We build T^* in stages. Inductively, we assume that $T \cup \{\theta_s\}$ is satisfiable and $\theta_{s+1} \models \theta_s$.

1. Stage 0: Let $\theta_0 = \exists x(x = x)$.
2. Stage s + 1 = 3i +1: At this stage, we ensure that the theory we are building is complete. If $T \cup \{\theta_s \cup \phi_i\}$ is satisfiable, we let $\theta_{s+1} = \theta_s \wedge \phi_i$. Otherwise, we let $\theta_{s+1} = \theta_s \wedge \neg\phi_i$.
3. Stage s+ 1 = 3i +2: At this stage, we ensure that the theory we are building has witnesses in C. If ϕ_i is of the form $\exists x\psi(x)$, and $\theta_s \models \phi_i$, let $c \in C$ be a constant not occuring in θ_s. Then we set $\theta_{s+1} = \theta_s \wedge \psi(c)$. Otherwise, we $\theta_{s+1} = \theta_s$. We claim that $T \cup \{\theta_{s+1}\}$ remains satisfiable.
4. Stage s+1 =3i +3: At this stage, we ensure that the canoincal model of T^* will be atomic. Let n be the smallest number such that all constants from C which appear in θ_s are among $\{c_0, ..., c_n\}$. Let $\psi(x_0, ..., x_n)$ be the $\mathcal{L}$-formula such that $\theta_s = \psi(c_0, ..., c_n)$. By our hypothesis, we have that $T \cup \{\theta_s\}$ is satisfiable. Since the isolated types are dense, this implies that there exists

some isolated type $p \in S_{n+1}(T)$ such that $p \in [\psi(x_0, ..., x_n)]$. Then there exists a formula $\chi(x_0, ..., x_n)$ which isolates p. We set $\theta_{s+1} = \chi(c_0, ..., c_n)$. We claim that $T \cup \{\chi(\bar{c})\}$ is satisfiable and that $\theta_{s+1} \models \theta_s$.

We claim that the canonical model, $C/\sim$, is atomic. □

Theorem 0.2. *Suppose that T is a complete theory in a countable language, and $A \subseteq M \models T$ is countable. If $|S_n(A)| < 2^{\aleph_0}$, then*

(1) the isolated types in $S_n(A)$ are dense.
(2) $|S_n(A)| \leq \aleph_0$.

In particular, if $|S_n(T)| \leq 2^{\aleph_0}$ for all $n \geq 1$, then T has a prime model.

Proof. We prove that if the isolated types are not dense in $|S_n(A)|$, then $|S_n(A)| \geq 2^{\aleph_0}$. The proof of (ii) is similar in flavor. The *in particular* portion follows from (i) and the previous theorem.

Suppose that the isolated types are not dense in $S_n(A)$. There eixsts exists a formula $\phi(\bar{x})$ such that $\phi(\bar{x})$ contains no isolated types. Then there exists an $\mathcal{L}(A)$-formula $\psi(\bar{x})$ such that $[\phi(\bar{x}) \wedge \psi(\bar{x})]$ and $[\phi(\bar{x}) \wedge \neg\psi(\bar{x})]$ are both non-empty. Again, both $[\phi(\bar{x}) \wedge \psi(\bar{x})]$ and $[\phi(\bar{x}) \wedge \neg\psi(\bar{x})]$ do not contain any isolated types.

We now build a binary tree of formulas. For each $\sigma \in 2^{<\omega}$, we have a collection of formulas such that

(1) Each $\phi_\sigma(\bar{x})$ is non-empty and contains no isolated types.
(2) If σ is an initial segment of τ, then $[\psi_\tau(\bar{x})] \subset [\psi_\sigma(\bar{x})]$.
(3) For each σ, $\psi_{\sigma i}(\bar{x}) \models \neg\psi_{\sigma i-1}(\bar{x})$.

We note that the processes before the bullet points above can be iterated since by replacing $[\psi(\bar{x})]$ with $[\phi(\bar{x}) \wedge \boldsymbol{\psi}(\bar{x})]$ and repeating this argument. This is how the tree of formulas is constructed. For each $f : \omega \to \{0, 1\}$, we have a partial type

$$p_f = \{\psi_{f(0),...,f(n)} : n \in \omega\}$$

We can complete p_f to a complete type, $\bar{p}_f \in S_n(A)$. We claim the map $f \to \bar{p}_f$ is injective which proves the claim. □

1. Homogeneous models and partial elementary maps

Definition 1.1. Let M, N be $\mathcal{L}$ structures and $B \subseteq M$. A map $f : B \to N$ is a partial elementary map if

$$M \models \phi(\bar{b}) \implies N \models \phi(f(\bar{b})),$$

For every $\mathcal{L}$-formula and finite tuple $\bar{b}$ from B.

Example 1.2. Suppose that $\bar{a} = (a_1, ..., a_n)$, $\bar{b} = (b_1, ..., b_n)$ and tuples in M^n such that $\text{tp}(\bar{a}/\emptyset) = \text{tp}(\bar{b}/\emptyset)$. Then the map $f : \{a_1, ..., a_n\} \to \{b_1, ..., b_n\}$ via $f(a_i) = b_i$ is a partial elementary map.

Definition 1.3. Let κ be an infinite cardinal. We say that $M \models T$ is κ-homogeneous is whenever $A \subseteq M$ with $|A| < \kappa$, $f : A \to M$ is a partial elementary map, and $a \in M$, then there exists a map $f* \supseteq f$ such that $f* : A \cup \{a\} \to M$ which is partial elementary. We say that M is homogeneous is M is $|M|$-homogeneous.

Proposition 1.4. *Suppose that M is homogeneous and $A \subseteq M$, $|A| < |M|$ and $f : A \to M$ is a partial elementary map. Then there exists an automorphism $\sigma : M \to M$ such that $\sigma \supseteq f$. In particular, if M is homogeneous and $\bar{a}, \bar{b} \in M^n$*

such that $\text{tp}(\bar{a}/\emptyset) = \text{tp}(\bar{b}/\emptyset)$*, then there exists an automorphism* $\sigma : M \to M$ *such that* $\sigma(\bar{a}) = \bar{b}$*.*

Proof. Enumerate model. Extend the function f one point at a time. Need to be a little careful so that at each step you add in each point in both the domain and range (by noticing that the inverse map is also partial elementary). Take union. □

Lemma 1.5. *If* M *is atomic then* M *is* $\aleph_0$*-homogeneous. If* M *is countable and atomic, then* M *is homogeneous.*

Proof. Suppose that $f : A \to B$ is a partial elementary map. Let $c \in M$. We have that $A = \{a_1, ..., a_n\}$, $B = \{b_1, ..., b_n\}$ and $f(a_i) = b_i$. It suffices to show that there exists some $d \in M$ such that $\text{tp}(\bar{a}c/\emptyset) = \text{tp}(\bar{b}d/\emptyset)$. Then the map $f \cup \{(c, d)\}$ is a partial elementary map.

Since M is atomic, we have that $\text{tp}(\bar{a}, c/\emptyset)$ is isolated. Hence there exists a formula $\theta(\bar{x}, y)$ which isolates this type. Hence $M \models \exists y \chi(\bar{a}, y)$. Since our map is partial elementary, we see that $M \models \exists y \chi(\bar{b}, y)$. Find $d \in M$ such that $M \models \chi(\bar{b}, d)$. Since χ isolates, we have that $\text{tp}(\bar{a}c/\emptyset) = \text{tp}(\bar{b}d/\emptyset)$. So we are done. □

Theorem 1.6. *Let* T *be a complete theory in a countable language. Suppose* M *and* N *are countable homogeneous models of* T *and* M *and* N *realizes the same types in* $S_n(T)$ *for* $n \geq 1$*. Then* $M \cong N$*.*

Proof. Back and forth argument. □

Corollary 1.7. *Let* T *be a complete theory in a countable language. If* M, N *are prime models of* T *then* $M \cong N$*.*

Proof. Since M and N are both prime, we know that both M and N are countable and atomic. Hence all the types of $S_n(T)$ which are realized in both M and N are precisely the isolated types. By the previous theorem, we conclude that $M \cong N$. □