PKU MODEL THEORY NOTES

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Theorem 0.1 (Morley's Categoricity Theorem)**.** *Suppose that T is a countable complete theory. Then T is $\aleph_1$-categorical if and only if T is κ-categorical for $\kappa > \aleph_1$.*

More generally, we can ask the following question.

Question 0.2. Give a countable complete theory T, what does the sprectrum function $I(T, -) : Cardinals \to Cardinals$ look like?

To answer these question, one needs to understand the complexity, combinatorics and configurations of definable sets.

The following conjecture is still open.

Conjecture 0.3 (Vaught's Conjecture)**.** *Suppose that $\aleph_1 \neq 2^{\aleph_0}$. Then it is known that $I(T, \aleph_0) \in \{1, 3, 4, 5, 6, ..., \aleph_0, \aleph_1, 2^{\aleph_0}\}$. Vaught's Conjecture is that it cannot be $\aleph_1$.*

1. Monster Models

Given a theory T, not all models of T see all of the complexity in the definable subsets of T. One way to see this complexity is to make sure we work in models which "have enough points".

Definition 1.1. We say that M is κ-saturated if for any $A \subseteq M$ such that $|A| < \kappa$, for every $p \in S_1(A)$, there exists some $b \in M$ such that $b \models p$. We say that M is saturated if M is $|M|$-saturated.

Proposition 1.2. *If M is saturated then M is homogeneous.*

Proof. Let $A \subseteq M$, $|A| < |M|$, $a \in M$, and $f : A \to B$ be a partial elementary map. We want to find a some d so that we can extend our function f to a partial elementary map $f^* : A \cup \{a\} \to B \cup \{d\}$ where $f^*|_A = f$ and $f(a) = d$. Consider

$$q = \{\varphi(x, f(\bar{a})) : \varphi(x, \bar{a}) \in \text{tp}(a/A)\} \in S_1(B).$$

Since M is saturated and $|A| = |B|$, there exists some $d \in M$ such that $d \models q$. Set $f^* = f \cup \{(a, d)\}$. □

Question 1.3. Do saturated models exist? Formally, not always in ZFC. However, there are several ways to develop a theory around this. We will assume an inaccessible cardinal to build saturated models which are quite large.

Definition 1.4. A Cardinal κ is called regular if it is uncountable and the cofinality of κ is κ, in symbols $cof(\kappa) = \kappa$. Recall that the cofinality is more or less how long it takes to get to the top of κ. Formally,

$$cof(\kappa) = \min\{|A| : A \subseteq \kappa, \sup(A) = \kappa\}.$$

Example 1.5. $\aleph_1, \aleph_2, ..., \alpha_{\omega+1}$, $\aleph_{\omega+2}$ are all regular cardinals. $\aleph_\omega$ is not a regular cardinal.

Definition 1.6. We say that a cardinal κ is strongly inaccessible if

(1) κ is a regular limit cardinal, i.e. $\kappa = \bigcup_{\mu<\kappa:\mu \text{ is a cardinal}} \mu$.
(2) For every $\mu < \kappa$, $2^\mu < \kappa$.

Proposition 1.7. *ZFC + "There exists a strongly inaccessible cardinal" proves there every countable theory has a saturated model.*

Proof. Let κ be a strongly inacessible cardinal. We build a model in κ many stages.

Stage 1: Let $M_1 = M$.

Stage $\alpha + 1$: Find a model $M_{\alpha+1}$ with the following properties:

(1) $M_\alpha \prec M_{\alpha+1}$.
(2) For every $p \in S_1(M_\alpha)$, there exists some $b \in M_{\alpha+1}$ such that $b \models p$.
(3) $|M_{\alpha+1}| = 2^{|M_\alpha|}$.

Stage γ for γ a limit ordinal: We let $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$. We remark that for any $\alpha < \gamma$, $M_\alpha \prec M_\gamma$.

Now, let $M_\kappa = \bigcup_{\alpha<\kappa} M_\alpha$. We claim that M_κ is a saturated model of T. Let $A \subset M_\kappa$ such that $|A| < \kappa$. We want to show that for every $p \in S_1(A)$, there exists some $b \in M_\kappa$ such that $b \models p$. Notice that there exists some $\alpha < \kappa$ such that $A \subseteq M_\alpha$. Otherwise, one can show that κ is not regular and has cofinality $|A|$. Now, we have that every type in $S_1(A)$ is realized in $M_{\alpha+1}$. So we are good. □

Definition 1.8. A monster model $\mathcal{U}$ is a saturated model of size κ where κ is a strongly inaccessible cardinal.

Remark 1.9. Monster models in model theory can be treated several different ways. In practice, we usually just choose a model which is μ-saturated for some large μ (larger than one we will ever think about). There, we do not need to strongly inaccessible assumption. Other authors actually make their monster model "class size".

2. Morley Rank

Definition 2.1. Fix a monster model $\mathcal{U}$ of T. We have a rank on all $\mathcal{L}(\mathcal{U})$-formulas, $\theta(\bar{x})$. In this section, I will often conflate a formula with its definable set. In partuclar, we will have

$$\theta(\bar{x}) = \theta(\mathcal{U}^{|\bar{x}|}) = \{\bar{a} \in \mathcal{U}^{|\bar{x}|} : \mathcal{U} \models \theta(\bar{a})\}.$$

Fix an $\mathcal{L}(\mathcal{U})$ formula $\theta(\bar{x})$.

(1) $RM(\theta(\bar{x})) \geq 0$ if $\theta(\bar{x})$ is non-empty.
(2) For any ordinal α, we have $RM(\theta(\bar{x})) \geq \alpha + 1$ if there exists a sequence of $\mathcal{L}_{\bar{x}}(\mathcal{U})$ forulas $(\psi_i(\bar{x}))_{i<\omega}$ with the following properties:
    (a) For each $i \in \omega$, $\psi_i(\bar{x}) \subseteq \theta(\bar{x})$.
    (b) For each $i \in \omega$, $RM(\psi_i(\bar{x})) \geq \alpha$.
    (c) For each $i, j \in \omega$ such that $i \neq j$, we have that $\psi_i(\bar{x}) \cap \psi_j(\bar{x}) = \emptyset$.
(3) For a limit ordinal γ, we have that $RM(\theta(\bar{x})) \geq \gamma$ if $RM(\theta(\bar{x})) \geq \alpha$ for each $\alpha < \gamma$.
(4) For any ordinal α, say that $RM(\theta(\bar{x})) = \alpha$ if $RM(\theta(\bar{x})) \geq \alpha$ and it is not the case that $RM(\theta(\bar{x})) \geq \alpha + 1$.

(5) We write that $RM(\theta(\bar{x})) = \infty$ if $RM(\theta(\bar{x})) \geq \alpha$ for every ordinal α.
(6) $RM(T) = RM(x = x)$.

Remark 2.2. Usually, it is easy to show that the rank of some formula is greater than some ordinal α. It is harder to show that the rank of a formula is bounded by some ordinal.

Fact 2.3. *Fix a monster model $\mathcal{U}$ of T and let $\theta(\bar{x})$ be an $\mathcal{L}(\mathcal{U})$ formula.*

(1) If $0 < |\theta(\mathcal{U}^{|\bar{x}|})| < \aleph_0$, then $RM(\theta(\bar{x})) = 0$.
(2) If $|\theta(\mathcal{U}^{|\bar{x}|})| \geq \aleph_0$, then $RM(\theta(\bar{x})) \geq 1$.

Example 2.4. Let T be the theory of infinitely many equivalence classes all with infinitely many elements. Then $RM(T) = 2$.

Example 2.5. $RM(DLO) = \infty$.