PKU MODEL THEORY NOTES

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Throughout these notes, x will denote a tuple of variables.

Lemma 0.1. *Suppose that* $\theta(x), \psi_1(x), \psi_2(x) \in \mathcal{L}_x(\mathcal{U})$. *If* $RM(\theta(x)) \geq \alpha$ *and* $\theta(x) \subseteq \psi_1(x) \cup \psi_2(x)$, *then* $RM(\theta(x) \wedge \psi_1(x)) \geq \alpha$ *or* $RM(\theta(x) \cap \psi_2(x)) \geq \alpha$.

Proof. Induction on α. □

Lemma 0.2. *Suppose that* $\psi_1(x), \psi_2(x) \in \mathcal{L}_x(\mathcal{U})$. *Then* $RM(\psi_1(x) \cup \psi_2(x)) = \max\{RM(\psi_1(x)), RM(\psi_2(x))\}$.

Definition 0.3. Let $\psi(x) \in \mathcal{L}_x(\mathcal{U})$ and suppose that $RM(\psi(x)) = \alpha$. Then the Morley degree of $\psi(x)$, denoted $dM(\psi(x))$, is the largest d such that there exists $\theta_1(x), ..., \theta_d(x) \in \mathcal{L}_x(\mathcal{U})$ such that

1. The θ_i's are pairwise disjoint, i.e. $\theta_i(x) \cap \theta_j(x) = \emptyset$ if $i \neq j$.
2. $RM(\theta_i(x)) = \alpha$ for each $i \leq d$.

The Morley degree of a theory is the Morley degree of $\mathbf{x} = \mathbf{x}$ where $\mathbf{x}$ is a singleton of variables. We write this as $dM(T)$.

Definition 0.4. We say that T is strongly minimal if $RM(T) = dM(T) = 1$.

Example 0.5. $\mathbf{ACF}_p$ and $\mathbf{ACF}_0$ are strongly minimal.

Lemma 0.6. *Suppose that* $A \subseteq \mathcal{U}$. *Let* $p \in S_x(A)$ *and suppose that* p *contains a ranked formula, i.e. there exists some* $\theta(x) \in p$ *such that* $MR(\theta(x))$ *is ordinal valued. Then* p *is completely determined by a single formula. Consider the formula* $\varphi_p(x) \in p$ *such that* $\varphi_p(x)$ *has the smallest Morley rank among all the formulas in* p, *say* α, *and out of all the formulas of rank* α *in* p, $\varphi_p(x)$ *has the smallest Morley degree. In other words, choose* $\varphi_p(x) \in p$ *such that for any* $\theta(x) \in p$, $RM(\varphi_p(x)) \leq RM(\theta(x))$ *and if* $RM(\varphi_p(x)) = RM(\theta(x))$, *then* $dM(\varphi_p(x)) \leq dM(\theta(x))$. *We claim that*

$$p = \{\psi(x) \in \mathcal{L}_x(A) : MR(\varphi_p(x) \backslash \psi(x)) < \alpha\}$$

Proof. Fix p and choose $\varphi_p(x)$ as above. Suppose that $MR(\varphi_p(x)) = \alpha$. We also assume for this proof that $dM(\varphi_p(x)) = 1$, just for simplicity.

First, suppose that $\psi(x) \in p$. Then $\varphi_p(x) \wedge \psi(x) \in p$. Notice that

1. $\varphi_p(x) \wedge \psi(x) \subseteq \varphi_p(x)$ and so $RM(\varphi_p(x) \wedge \psi(x)) \leq RM(\varphi_p(x)) = \alpha$. Notice that $\varphi_p(x) \wedge \psi(x)$ cannot have rank strictly less than α, otherwise $\varphi_p(x)$ would not be a minimal choice. Hence $RM(\varphi_p(x) \wedge \psi(x)) = \alpha$
2. Also, $\varphi_p(x) \wedge \neg\psi(x) \subseteq \varphi_p(x)$ and so $RM(\varphi_p(x) \wedge \neg\psi(x)) \leq RM(\varphi_p(x)) = \alpha$.

Now, if $RM(\varphi_p(x) \wedge \neg\psi(x)) = \alpha$, then $mD(\psi_p(x)) \geq 2$, which is a contradiction. Thus, $RM(\psi_p(x) \wedge \neg\psi(x)) < \alpha$.

Second, suppose that $RM(\varphi_p(x) \wedge \neg\psi(x)) < \alpha$. Since p is a type, either $\neg\psi(x) \in p$ or $\psi(x) \in p$. Notice that if $\neg\psi(x) \in p$, then $\varphi_p(x) \wedge \neg\psi(x) \in p$ and so $\varphi_p(x)$ is not minimal. Hence it must follow that $\psi(x) \in p$. □

Definition 0.7. Let T be a complete theory. For emphasis, we will use $\bar{x}$ to denote a tuple of variables.

(1) We say that T is totally transcendental if for every formula $\psi(\bar{x}) \in \mathcal{L}_{\bar{x}}(\mathcal{U})$, we have that $RM(\psi(\bar{x})) < \infty$.
(2) We say that T is ω-stable if for any $A \subset \mathcal{U}$, $|A| = \aleph_0$, $|S_{\bar{x}}(A)| = \aleph_0$.

Theorem 0.8. *Suppose that T is a countable complete theory. Then T is totally transcendental if and only if T is ω-stable.*

Proof. We first show the forward direction. Suppose that T is t.t.. The for any $p \in S_{\bar{x}}(A)$ there exists a formula $\varphi_p(\bar{x})$ such that

$$p = \{\psi(\bar{x}) \in \mathcal{L}_{\bar{x}}(A) : RM(\varphi_p(\bar{x}) \backslash \psi(\bar{x})) < RM(\varphi_p(x))\}.$$

The map $p \to \varphi_p$ is an injection from $S_{\bar{x}}(A)$ to $\mathcal{L}_{\bar{x}}(A)$, which implies $|S_{\bar{x}}(A)| \leq \aleph_0$.

Now we prove the backwards direction. Suppose that T is not totally t.t.. Then there exists a formula $\psi(\bar{x}) \in \mathcal{L}_{\bar{x}}(\mathcal{U})$ such that $RM(\psi(\bar{x})) = \infty$. [HW] Now, there exists $\psi_0(\bar{x})$ and $\psi_1(\bar{x})$ such that

(1) $\psi_0(\bar{x}), \psi_1(\bar{x}) \subset \psi(\bar{x})$.
(2) $\psi_0(\bar{x}) \cap \psi_1(\bar{x}) = \emptyset$.
(3) $RM(\psi_0(\bar{x})) = RM(\psi_1(\bar{x})) = \infty$

Now iterate and build an infinite binary tree of formulas. We claim there are $2^{\aleph_0}$ many types the parameters used in the formulas to construct this tree. □

1. Stability

Definition 1.1. A formula $\varphi(\bar{x}, \bar{y})$ is said to be stable if there do not exists sequence of points $(\bar{a}_i)_{i<\omega}$ and $(\bar{b}_j)_{j<\omega}$ such that

$$\mathcal{U} \models \varphi(\bar{a}_i, \bar{b}_j) \Longleftrightarrow i \leq j.$$

We say that $\varphi(\bar{x}, \bar{y})$ is k-stable if there does not exists $(\bar{a}_i, \bar{b}_j)_{1 \leq i,j \leq k}$ such that

$$\mathcal{U} \models \varphi(\bar{a}_i, \bar{b}_j) \Longleftrightarrow \leq i \leq j.$$

We say that T is stable if every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ is stable.

Proposition 1.2. *If T is ω-stable, then T is stable.*

Proof. Suppose that T is unstable. Then there exists a formula $\varphi(\bar{x}, \bar{y})$ and sequences $(\bar{a}_i)_{i<\omega}$ and $(\bar{b}_j)_{j<\omega}$ witnessing the order property. By compactness, there exists $(\bar{c}_i, \bar{d}_j)_{(i,j) \in \mathbb{Q} \times \mathbb{Q}}$ such that

$$\mathcal{U} \models \varphi(c_i, d_j) \Longleftrightarrow i \leq j.$$

Then, for each $r \in \mathbb{R} \backslash \mathbb{Q}$, we let p_r be a complete type such that

$$p_r \subseteq \{\neg\varphi(\bar{x}, \bar{d}_k) \wedge \varphi(\bar{x}, \bar{d}_j) : k < r < j\}$$

Then we have an injection from $\mathbb{R} \backslash \mathbb{Q} \to S_{\bar{x}}(D)$ where D is the collection of parameters occurs in $(\bar{d}_j)_{j \in \mathbb{Q}}$. □

Example 1.3. (1) 'x divides y' in $\mathbb{N}$ is unstable.
(2) Let G be a group and H be a subgroup. Then the relation $xH = yH$ is 2-stable.