PKU MODEL THEORY NOTES

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[Note: Most of this section comes from Artem Chernikov's Introduction to stability notes.]

The definition of a type works in infinite many variables, not just finitely many. Let $(x_i)_{i \in I}$ be a possibly infinite collection of variables. If $A \subseteq M$, then a complete type $p \in S_{(x_i)_{i\in I}}(A)$ is a subset of $\mathcal{L}_{(x_i)_{i \in I}}(A)$ which is both finitely satisfiable and for each formula $\psi(x_{i_1},...,x_{i_n}) \in \mathcal{L}_{(x_i)_{i\in I}}(A)$, either $\psi(x_{i_1},...,x_{i_n}) \in p$ or $\neg\psi(x_{i_1},...,x_{i_n}) \in p$.

First we recall the definition of an indiscernible sequence.

Definition 0.1. Let I be an indexing set and $(\bar{a}_i)_{i \in I} \in \mathcal{U}^n$ be a sequence of elements. We say that $(\bar{a}_i)_{i\in I}$ is indiscernible (over B) if for every increasing sequence of indices $i_1 < .... < i_n$ and $j_1 < ... < j_n$ and formula $\varphi(x_1,...,x_n) \in \mathcal{L}(B)$,

$$\mathcal{U} \models \varphi(a_{i_1},...,a_{i_n}) \leftrightarrow \varphi(a_{j_1},...,a_{j_n}).$$

Moreover, we say that the sequence is totally indiscernible if for any collection of indices $i_1 \neq ... \neq i_n$ and $j_1 \neq .... \neq j_n$ we have that

$$\mathcal{U} \models \varphi(a_{i_1},...,a_{i_n}) \leftrightarrow \varphi(a_{j_1},...,a_{j_n}).$$

Remark 0.2. We say that a sequence $(\bar{a}_i)_{i\in I}$ is indiscernible/totally indiscernible if it is indiscernible/totally indiscernible over $\emptyset$.

Definition 0.3. Suppose that $(\bar{a}_i)_{i\in I}$ is any sequence from $\mathcal{U}^n$ and B is a set of parameters (i.e. $B \subseteq \mathcal{U}$). Then the EM-type, or the *Ehrenfeucht-Mostowski type* of the sequence $(\bar{a}_i)_{i\in I}$ (over B) is the partial type indexed by ω given by

$$\{\varphi(x_0,...,x_n) \in \mathcal{L}_{(x_i)_{i<\omega}}(B) : \forall i_0 < ... < i_n, \mathcal{U} \models \varphi(a_{i_0},...,a_{i_n})\}.$$

We let $\mathrm{EM}((\bar{a}_i)_{i\in I}/B)$ denote the EM-type of $(\bar{a}_i)_{i\in I}$ over B.

Remark 0.4. Suppose that $(\bar{a}_i)_{i\in I}$ is B-indiscernible sequence. Then $\mathrm{EM}((\bar{a})_{i\in I}/B) \in S_{(x_i)_{i<\omega}}(B)$. In other words, it is a complete type.

Proposition 0.5. *Let I be a (small) infinite indexing set and $(\bar{a}_i)_{i\in I} \in \mathcal{U}^n$ be a sequence of elements. Let B be a (small) set of parameters. Then for any other (small) infinite indexing set J there exists elements $(\bar{b}_j)_{j\in J}$ such that*

1. *$(\bar{b}_j)_{j\in J} \models \mathrm{EM}((\bar{a}_i)_{i\in I}/B)$.*
2. *$(\bar{b}_j)_{j\in J}$ is B-indiscernible.*

Proof. Compactness + Ramsey. □

Corollary 0.6. *Let B be a (small) set of parameters. Let I be a (small) infinite indexing set and $(\bar{a}_i)_{i\in I} \in \mathcal{U}^n$ be a B-indiscernible sequence. Let $J \subseteq I$ where the size of J is small. Then there exists a sequence $(\bar{b}_j)_{j\in J}$ such that*

1. *$(\bar{b}_j)_{j\in J}$ is B-indiscernible.*
2. *$a_i = b_i$ for each $i \in I$.*

Proof. By the previous proposition, there exists a sequence $(c_j)_{j\in J}$ such that

(1) $\text{EM}((\bar{c}_j)_{j\in J}/B) = \text{EM}((\bar{a}_i)_{i\in I}/B)$.
(2) $(\bar{c}_j)_{j\in J}$ is B-indiscernible.

This implies that $\text{tp}((\bar{c}_j)_{j\in J}/B) = \text{tp}((\bar{c}_j)_{j\in J}/B)$. Hence there exists an automorphism σ of $\mathcal{U}$ (fixing B) such that $\sigma(c_i) = a_i$. Then the sequence $(\sigma(c_j))_{j\in J}$ works. □

1. Stability and indiscernibles

Proposition 1.1. *The following are equivalent*

(1) T is stable.
(2) There is no sequence of tuples $(\bar{a}_i)_{i\in I}$ for $\mathcal{U}^n$ and formula $\varphi(\bar{x}_1, \bar{x}_2) \in \mathcal{L}(\mathcal{U})$ such that

$$\mathcal{U} \models \varphi(\bar{a}_i, \bar{a}_j) \Longleftrightarrow i \leq j.$$

Proof. One direction is trivial. Now suppose that T is unstable. Then there exists a formula $\varphi(\bar{x}, \bar{y})$ and sequence $(\bar{a}_i, \bar{b}_j)_{i,j\in\omega}$ such that

$$\mathcal{U} \models \varphi(\bar{a}_i, \bar{b}_j) \Longleftrightarrow i \leq j.$$

Consider the new formula given by $\theta(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2) := \varphi(\bar{x}_1, \bar{y}_2) \wedge \bar{x}_2 = \bar{x}_2 \wedge \bar{y}_1 = \bar{y}_1$. Then the sequence $(\bar{c}_i)_{i\in\omega}$ given by $\bar{c}_i = (\bar{a}_i, \bar{b}_i)$ with the formula $\theta(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)$ works. □

Theorem 1.2. *The following are equivalent:*

(1) T is stable.
(2) Every indiscernible sequence is totally indiscernible.

Proof. First suppose that T is unstable. Then there exists a sequence $(\bar{a}_i)_{i\in\omega}$ and a formula $\varphi(\bar{x}, \bar{y}, \bar{d})$ such that

$$\mathcal{U} \models \varphi(\bar{a}_i, \bar{a}_j, \bar{d}) \Longleftrightarrow i \leq j.$$

Choosing $B = \{d_i : \bar{d} = (d_1, ..., d_n)\}$, there exists an B-indiscernible sequence $(\bar{c}_i)_{i\in\omega}$ such that

(1) $(\bar{c}_i)_{i\in\omega} \models \text{EM}(\bar{a}_i/B)$.
(2) $(\bar{c}_i)_{i\in\omega}$ is B-indiscernible.

Then $\mathcal{U} \models \varphi(\bar{c}_1, \bar{c}_3, \bar{d})$ and $\mathcal{U} \models \neg\varphi(\bar{c}_3, \bar{c}_1, \bar{d})$. Hence our sequence is not totally indiscernible.

Now suppose that $(\bar{a}_i)_{i\in I}$ is indiscernible but not totally indiscernible. WLOG, we may assume that $I = \mathbb{Q}$. So there exists some formula $\theta(x_1, ..., x_n) \in \mathcal{L}(A)$ and indices $r_1 < ... < r_n$ and some $\sigma \in Sym(n)$ such that $\mathcal{U} \models \varphi(\bar{a}_{r_1}, ..., \bar{a}_{r_n}) \wedge \neg\varphi(\bar{a}_{r_{\sigma(1)}}, ..., \bar{a}_{r_{\sigma(n)}})$. Since $\sigma \in Sym(n)$, we can write $\sigma = \prod_{1\leq j\leq k} \tau_i$ where each τ_i is a transposition of two consecutive elements. We let $\sigma_j = \prod_{1\leq i\leq j} \tau_i$. Let j be the smallest integer such that

$$\mathcal{U} \models \varphi(\bar{a}_{r_{\sigma_{j-1}(1)}}, ..., \bar{a}_{r_{\sigma_{j-1}(n)}}) \wedge \neg\varphi(\bar{a}_{r_{\sigma_j(1)}}, ..., \bar{a}_{r_{\sigma_j(n)}})$$

which used transposition $\tau_j = (s, s+1)$ for some $1 \leq s < n$. Then consider the formula

$$\psi(x_1, ..., x_n) := \varphi(x_{\sigma_{j-1}(1)}, ..., x_{\sigma_{j-1}(n)}).$$

and $\theta(x_1, x_n) := \psi(\bar{a}_{r_1}, ..., \bar{a}_{r_{s-1}}, x_1, x_2, \bar{a}_{r_{s+1}}, ..., \bar{a}_{r_n})$. Then this defines an order on any sequence from the interval $\{\bar{a}_i : r_{s-1} < i < r_{s+2}\}$. □

Theorem 1.3. *Let $\varphi(\bar{x}, \bar{y})$ be any stable formula, from any theory. Then there exists some k depending only on φ such that for any indiscernible sequence $(\bar{a}_i)_{i \in I}$ from $\mathcal{U}^{\bar{x}}$ and $\bar{b} \in \mathcal{U}^{\bar{y}}$, either*

$$|\psi(A, b)| \leq k \text{ or } |\neg\psi(A, b)| \leq k,$$

where $A = \{\bar{a}_i : i \in I\}$.

Proof. Let $\varphi(\bar{x}, \bar{y})$ be stable. Then there exists some k such that $\varphi(\bar{x}, \bar{y}, \bar{d})$ is k-stable. This will be our choice of k. Let $(\bar{a}_i)_{i \in I}$ be an indiscernible sequence and $\bar{b}$ be a parameter. WLOG, we may assume that $I = \mathbb{N}$. We claim that either

$$|\psi(A, b)| \leq k \text{ or } |\neg\psi(A, b)| \leq k.$$

Suppose not. Then

$$|\psi(A, b)| > k \text{ or } |\neg\psi(A, b)| > k.$$

So there exists indices $i_1, ..., i_{k+1}, j_1, ..., j_{k+1}$ such that

$$\mathcal{U} \models \bigwedge_{1 \leq l \leq k+1} \neg\varphi(\bar{a}_{i_l}, \bar{b}) \wedge \bigwedge_{1 \leq t \leq k+1} \varphi(\bar{a}_{j_t}, \bar{b}).$$

Total indiscernibility plus an autormophism argument implies there exists some $\bar{b}'$ such that

$$\mathcal{U} \models \bigwedge_{1 \leq l \leq k+1} \neg\varphi(\bar{a}_l, \bar{b}') \wedge \bigwedge_{k+2 \leq t \leq 2k+2} \varphi(\bar{a}_t, \bar{b}').$$

Then,

$$\mathcal{U} \models \exists \bar{y} \bigwedge_{1 \leq l \leq k+1} \neg\varphi(\bar{a}_l, \bar{y}) \wedge \bigwedge_{k+2 \leq t \leq 2k+2} \varphi(\bar{a}_t, \bar{y}).$$

By indiscerniblility, for each $1 \leq t \leq k+1$,

$$\mathcal{U} \models \exists \bar{y} \bigwedge_{1 \leq l \leq t} \neg\varphi(\bar{a}_l, \bar{y}) \wedge \bigwedge_{t < k+1} \varphi(\bar{a}_t, \bar{y}).$$

and so for each $1 \leq t \leq k+1$, there exists some $\bar{b}_t$ such that

$$\mathcal{U} \models \bigwedge_{1 \leq l \leq t} \neg\varphi(\bar{a}_l, \bar{b}_t) \wedge \bigwedge_{t < k+1} \varphi(\bar{a}_t, \bar{b}_t).$$

But this implies that $\varphi(x, y)$ is not k-stable.

□