PKU MODEL THEORY NOTES

KYLE GANNON

[Portions of these notes come from the monograph *Model theory and Algebraic geometry*; specifically Zeigler and Lascar's sections]

Question 0.1. *Suppose we are in a totally transcendental theory. Let $A \subseteq B \subseteq \mathcal{U}$ and $p \in S_x(A)$. What is the "most generic" extension of p to a type in $S_x(B)$?*

Example 0.2. Consider the theory of infinitely many equivalence classes all with infitely many elements. Let $M \subset N \subset \mathcal{U}$ be models of our theory. Consider the unique complete type $p \in S_x(M)$ where $p \subseteq \{\neg xEa : a \in M\}$. Now consider three points;

(1) $b_1 \in N$. This gives us the type $q_1 = \mathrm{tp}(b_1/N)$.
(2) $b_2 \notin N$ but there exists $c \in N$ such that cEb. This gives us the type $q_2 = \mathrm{tp}(b_2/N)$
(3) $b_3 \notin N$ and there does not exists any $c \in N$ such that cEb. This gives us the type $q_3 = \mathrm{tp}(b_3/N)$.

The types q_1, q_2 and q_3 each extend the type p. However, q_3 is the most generic extension. No new information is given about the type. In particular, there is no drop in rank.

Definition 0.3. Fix $A \subseteq \mathcal{U}$ and let $p \in S_x(A)$.

(1) Then the Morley rank of p, denoted $MR(p)$ is $\inf\{MR(\theta(x)) : \theta(x) \in p\}$.
(2) Then the Morley degree of p, denoted $Md(p)$ is $\inf\{Md(\theta(x)) : \theta(x) \in p, MR(\theta(x)) = MR(p)\}$.

Proposition 0.4. *Suppose that T is totally transcendental. If $\theta(x) \in \mathcal{L}_x(B)$ then there exists some $p \in S_x(B)$ such that*

(1) $\theta(x) \in p$.
(2) $MR(\theta(x)) = MR(p)$.

Proof. Suppose not. Let $MR(\theta(x)) = \alpha$. Consider the set $[\theta(x)]_B = \{q \in S_x(B) : \theta(x) \in q\}$. This set is compact. Now for each $p \in [\theta(x)]_B$, there exists some $\psi_p(x) \in p$ such that $MR(\psi_p(x)) < \alpha$. Then

$$\bigcup_{p \in [\theta(x)]_B} [\psi_p(x)]_B$$

is an open cover of $[\theta(x)]_B$. Thus there is a finite subcover, say $[\theta(x)]_B \subseteq [\psi_{p_1}(x)]_B \cup ... \cup [\psi_{p_k}(x)]_B$. But then

$$MR(\theta(x)) \leq MR(\psi_{p_1}(x) \cup ... \cup \psi_{p_k}(x)) = \max\{MR(\psi_{p_1}(x)), ..., MR(\psi_{p_k}(x))\} < \alpha.$$

□

Fact 0.5. *If T is totally transcendental, then*

$$0 < |\{p \in S_x(\mathcal{U}) : MR(p) = MR(T)\}| < \alpha_0.$$

Definition 0.6. Let $A \subseteq B \subseteq \mathcal{U}$. Let $p \in S_x(A)$, $q \in S_x(B)$, and $p \subseteq q$. We say that 'q is a non-forking extension of p' if $MR(q) = MR(p)$.

Proposition 0.7. *Let T be a totally transcendental theory. Let $A \subseteq B \subseteq \mathcal{U}$ and $p \in S_x(A)$.*

(1) There exists some $q \in S_x(B)$ such that q is a non-forking extension of p.
(2) $|\{q \in S_x(B) : q \text{ is a non-forking extension of } p\}| \leq Md(p)$. More explicitly, we have that $Md(p) = \sum_{q \in F} Md(q)$ where F is the collection of non-forking extension of p in $S_x(B)$.
(3) If $A = M \models T$, then $Md(p) = 1$. Hence if $p \in S_x(M)$ and $M \subseteq B$, there exists a unique non-forking extension of p in $S_x(B)$.

Proof. Exercise. For (1), use the fact that p is completely determined by a single formula. □

Definition 0.8. We say that a is free from B over C is the type $\text{tp}(a/BC)$ is a non-forking extension of $\text{tp}(a/C)$. We write,

$$a \mathop{\smile\!\!\!\!\!\!|}_{C} B$$

Moreover, we write $A \mathop{\smile\!\!\!\!\!\!|}_{C} B$ if for every finite tuple $\bar{a}$ from A, $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{C} B$.

Example 0.9. We give some basic examples

(1) In $\mathbf{ACF}_0$, $\pi \mathop{\smile\!\!\!\!\!\!|} e$ while $\pi + e \not\mathop{\smile\!\!\!\!\!\!|}_{\{\pi\}} e$.
(2) If V is an infinite dimensional vector space over the field K in the usual language, then $v \not\mathop{\smile\!\!\!\!\!\!|}_{C} B$ if and only if v is in the K-span of $B \cup C$ and not in the span of C.

1. Omega-stable groups

Let T be a theory in a language $\mathcal{L}$ which extends the language of groups. We let G be a monster model of T.

Proposition 1.1. *If H is a definable subgroup of G, then for any $a \in G$, we have that $MR(H) = MR(aH) = MR(Ha)$.*

Proof. Morley Ranks is preserved under definable bijections. □

Corollary 1.2. *Assume that $H \subsetneq H'$ are definable subgroups of G. Then*

(1) If $[H' : H]$ is finite, then $MR(H) = MR(H')$ and $Md(H) \cdot |H'/H|$.
(2) If $[H' : H]$ is infinite, then $MR(H') > MR(H)$.

Corollary 1.3. *There is no infinite decreasing sequence of definable subgroups.*

Corollary 1.4. *The intersection of any class of definable subgroups is equal to the intersection of a finite number and is thus definable.*

Definition 1.5. Let H be a definable subgroup of G. We say that H is definably connected if it contains no proper subgroup of finite index.

Definition 1.6. The connected component of G is the intersection of all definable subgroups of G of finite index. It is denoted G^0.

Remark 1.7. We know the following:

(1) G^0 is a definable subgroup of G of finite index.
(2) G^0 is connected.
(3) $MR(G) = MR(G^0)$.
(4) G^0 is normal.

We will work toward proving the following theorem:

Theorem 1.8. *Suppose that T is a totally transcendental theory. Then*

$$G/G^0 \cong (\{p \in S_x(G) : MR(p) = MR(T)\}, *)$$

where $p * q = \mathrm{tp}(a \cdot b/G)$ where $b \models q$, $a \models p'|_{Gb}$ where p' is the unique non-forking extension of p.

Definition 1.9. Let $\mathbf{G}$ be any group. Then a set $X \subset \mathbf{G}$ is said to be generic if finitely many translates cover the group.

Example 1.10. If we consider $\mathbb{Z}$, then $2\mathbb{Z}$ is generic while $\mathbb{N}$ is not.

Definition 1.11. Let $p \in S_x(G)$. We say that p is generic if for every formula $\psi(x) \in p$, the set $\psi(G) = \{g \in G : G \models \psi(g)\}$ is generic.

Proposition 1.12. *Suppose that T is totally transcendental. If $p \in S_x(G)$ and p is generic, then p has maximal rank.*

Proof. Exercise. $\square$