PKU MODEL THEORY NOTES

KYLE GANNON

1. Definability of types

[Portions of these notes come from Markers "Model theory: An introduction"]. T will always be from a countable language.

Definition 1.1. Let $A \subseteq B \subseteq \mathcal{U}$ and $p \in S_x(B)$. We say that p is definable (over A) if for any formula $\varphi(x, \bar{y})$ there exists a formula $d_p^{\varphi}(\bar{y}) \in \mathcal{L}_y(A)$ such that

$$\varphi(x, \bar{b}) \in p \longleftrightarrow \mathcal{U} \models d_{\varphi}^{p}(\bar{b}).$$

In ω-stable theories, all types are definable.

Theorem 1.2. *Suppose that T is ω-stable. Let $\mathcal{M} \prec \mathcal{U}$ be an $\aleph_0$-saturated model of T and $\varphi(\bar{x})$ be an $\mathcal{L}(\mathcal{M})$-formula with $RM(\varphi(\bar{x})) = \alpha$ and $\psi(x, y)$ be an $\mathcal{L}$-formula. Then the set*

$$\{\bar{b} \in \mathcal{U} : RM(\varphi(\bar{x}) \wedge \psi(x, \bar{b})) = \alpha\}$$

is definable with parameters from $\mathcal{M}$.

Corollary 1.3. *Suppose that T is ω-stable. Let $A \subseteq \mathcal{U}$ and $p \in S_x(A)$. Then there exists some A_0, a finite subset of A, such that p is definable over A_0.*

Proof. Use the previous theorem and the fact taht types in ω-stable theories are completely controlled by a single formula (the one of minimal rank and degree). □

2. Groups

Throughout these notes, let G be a monster model of a group.

Definition 2.1. Let G be a group and $A \subseteq G$. Then the centralizer of A is

$$C(A) = \bigcap_{a \in A} C(\{a\})$$

where $C(a) = \{g \in G : ga = ag\}$.

Proposition 2.2. *Suppose that G is ω-stable. Then there exists $a_1, ..., a_n \in A$ such that $C(A) = \{g \in G : ga_i = a_i g$ for all $i = 1, ...n\}$.*

Proof. Intersection of infinitely many groups is equal to a finite intersection. □

Proposition 2.3. *Let G be ω-stable. Then G^0 is $\emptyset$-definable.*

Proof. Let $G^0 = \varphi(x, \bar{b})$ for some $\bar{b}$ and $[G : G^0] = n$. We let $W = \{\bar{a} : \varphi(x, \bar{a})$ defines a subgroup of index $n\}$. Now, if $\bar{c} \in W$, consider $H = \{g \in G : G \models \varphi(g, \bar{c})\}$. Then, $H \cap G^0$ is a finite index subgroup of G^0 (and so finite index in G. Since G^0 is the smallest definable subgroup of finite index, we have that $H = G^0$. Then, $G^0 = \{g \in G : \exists \bar{y} \in W \wedge \varphi(g, \bar{y})\}$. □

Remark 2.4. The previous proof shows that G^0 is the unique subgroup of largest (finite) index.

Proposition 2.5. *Let G be an ω-stable group. Then G^0 is normal.*

Proof. If $h \in G$, then the map $x \to hxh^{-1}$ is a group automorphism. then hG^0h^{-1} is a definable subgroup of G. Moreover, we have that $[G : hG^0h^{-1}] = [G : G^0]$. By the previous remark, we have that $hG^0h^{-1} = G^0$ and so G^0 is normal. □

2.1. **Stabilizers.**

Definition 2.6. We have a natural action of G of $S_1(G)$ given by $g \cdot p = \{\varphi(x) : \varphi(g \cdot x) \in p\}$.

Definition 2.7. The stabilizer of p, denoted $\mathrm{Stab}(p)$, is $\{g \in G : g \cdot p = p\}$.

Theorem 2.8. *Suppose that G is ω-stable. Then* $\mathrm{Stab}(p)$ *is a definable group.*

Proof. For any $\mathcal{L}_x(G)$ formula $\varphi(x)$ we let $\mathrm{Stab}^{\varphi}(p) := \{g \in G : \forall h \in G, \varphi(h \cdot x) \in p$ if and only if $\varphi(h \cdot g \cdot x) \in p\}$. We claim that

(1) $\mathrm{Stab}(p) = \bigcap_{\varphi \in p} \mathrm{Stab}^{\varphi}(p)$.
(2) For each $\varphi(x) \in p$, $\mathrm{Stab}^{\varphi}(p)$ is a definable subgroup of G. Definability follows from the fact that the type p is defianble. For any formula $\varphi(x) \in p$, consider the formula $\varphi(x \cdot y)$. Since p is definable, we claim that $\mathrm{Stab}^{\varphi}(p) = \forall y(d^p_{\varphi}(y) \leftrightarrow d^p_{\varphi}(y \cdot x))$.
(3) Hence $\mathrm{Stab}(p)$ is the intersection of infinitely many definable groups. Thus it is the intersection of finitely many of them and hence definable.

□

Fact 2.9. *If $g \in G \prec G'$, there exists $a, b \in G'$ such that*

(1) $RM(\mathrm{tp}(a/G)) = RM(\mathrm{tp}(b/G)) = RM(T)$.
(2) $b \cdot a = g$.

Proof. Let $p \in S_1(G)$ have maximal rank. Choose $a \models p$ and consider $b = g \cdot a^{-1}$. We claim this works. □

Fact 2.10. *G has a unique generic type if and only if G is definably connected.*

Corollary 2.11. *Suppose that G is definably connected and A be a definable subset of G. If $RM(A) = RM(G)$, then $G = A \cdot A$.*

Proof. Let $\psi(x)$ be a definition for A. Consider $G \prec G'$. For every $g \in G$, there exists generic $a, b \in G'$ such that $a \cdot b = g$. Since G is definably connected, there is a unique generic type. Hence $G' \models \psi(a) \wedge \psi(b)$. Thus for any $g \in G$, we have that $G' \models \exists x \exists y(\psi(a) \wedge \psi(b) \wedge x \cdot y = g)$. Since $G \prec G'$, it follows that $G \models \exists x \exists y(\psi(a) \wedge \psi(b) \wedge x \cdot y = g)$, but this implies that there exists $a', b' \in A$ such that $a' \cdot b' = g$. □