PKU MODEL THEORY: LECTURE 2

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Definition 0.1. Let Σ be an $\mathcal{L}$-theory. Σ is inconsistent if $\Sigma \vdash \varphi$ for every $\mathcal{L}$-sentence φ. Otherwise, we say that Σ is consistent.

Proposition 0.2. *Let $A \in \mathcal{L}$ and Σ be an $\mathcal{L}$-theory. Then Σ is inconsistent if and only if $\Sigma \vdash (A \wedge \neg A)$.*

Proof. The forward direction is trivial. Let φ be any $\mathcal{L}$ sentence. Let $\theta_1, ..., \theta_n$ be a proof of $(A \wedge \neg A)$ from Σ. Notice that $(A \wedge \neg A) \to \varphi$ is valid. We claim that $\theta_1, ..., \theta_n, (A \wedge \neg A) \to \varphi, \varphi$ is a proof of φ from Σ. □

1. Completeness and Compactness

Theorem 1.1 (Completeness Theorem)**.** *Σ is consistent if and only if there exists M such that $M \models \Sigma$.*

Theorem 1.2 (Compactness Theorem)**.** *Σ is consistent if and only if for any $\Sigma_0 \subseteq \Sigma$ such that $|\Sigma_0|$ is finite, there exists M_0 such that $M_0 \models \Sigma_0$.*

Proposition 1.3. *If Σ is consistent then $\Gamma = \{\varphi : \Sigma \vdash \varphi\}$ is consistent. Γ is called the deductive closure of Σ.*

Proof. Suppose that Γ is inconsistent. In particular, $\Gamma \vdash (A \wedge \neg A)$ for some $A \in \mathcal{L}$. Let $\theta_1, ..., \theta_n$ be a proof of φ from Γ. By definition, we know that for each $i \leq n$, either

(1) θ_i is valid.
(2) $\theta_i \in \Gamma$.
(3) θ_i is inferred by two previous sentences.

Notice that if $\theta_i \in \Gamma$ and $i < n$, then $\Sigma \vdash \theta_i$. Hence there exists $\chi_{i_1}, ..., \chi_{i_m}$ which is a proof of θ_i from Σ. In the proof $\theta_1, ..., \theta_n$, replace each $\theta_i \in \Gamma \backslash \Sigma$ with $\chi_{i_1}, ..., \chi_{i_m}$. We claim that this new sting of sentences is a proof of φ from Σ. □

Definition 1.4. Σ is said to be *maximally consistent* if Σ is consistent and there does not exists $\Sigma' \supsetneq \Sigma$ such that Σ' is consistent.

Example 1.5. Let M be a $\mathcal{L}$-model. Then $\{\varphi : M \models \varphi\}$ is maximally consistent.

Proposition 1.6. *If Σ is maximally consistent and $\Sigma \vdash \varphi$. Then $\varphi \in \Sigma$.*

Proof. Suppose $\varphi \notin \Sigma$. Since Σ is maximally consistent, $\Sigma \cup \{\varphi\}$ is inconsistent. Notice that $\Sigma \cup \{\varphi\} \subseteq \{\varphi : \Sigma \vdash \varphi\}$ and if $\Sigma \cup \{\varphi\}$ is inconsistent, then so is $\{\varphi : \Sigma \vdash \varphi\}$. By Proposition 1.3, this implies that Σ is inconsistent and so we have a contradiction. □

Proposition 1.7 (Deduction Theorem)**.** *If $\Sigma \cup \{\psi\} \vdash \varphi$, then $\Sigma \vdash \psi \to \varphi$.*

Proof. Exercise. □

1.1. **Zorn's lemma.** If we are given a consistent theory, it is useful to extend to a maximally consistent theory. To do so, we need to use Zorn's lemma.

Definition 1.8. A partial order is a set P with a binary relation $\leq$ which is reflexive, anti-symmetric, and transitive.

(1) Reflexive: For any $x \in P$, $x \leq x$.
(2) Anti-symmetric: For any $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.
(3) Transitive: For any $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Example 1.9. $(\mathbb{N}, \leq)$ is a partial order. $(\mathcal{P}(\mathbb{N}), \subseteq)$ is also a partial order.

Definition 1.10. Let $(P, \leq)$ be a partial order.

(1) A chain is a subset of P which is totally ordered, i.e. C is a chain if $C \subset P$ and for any x, y in C, $x \leq y$ or $y \leq x$.
(2) A chain C has an upper bound if there exists some $a \in P$ such that for any $x \in C$, $x \leq a$.
(3) An element $m \in P$ is called maximal if there does not exists some $x \in P$ such that $m \leq x$ and $m \neq x$.

We now give the statement of Zorn's lemma:

Lemma 1.11 (Zorn's Lemma)**.** *Let $(P, \leq)$ be a partial order. Suppose that for every chain C of P, C has an upper bound. Then P contains at least one maximal element.*

Theorem 1.12 (Lidenbaum's Theorem)**.** *Let Σ be a consistent $\mathcal{L}$ theory. Then there exists a maximally consistent theory Σ' such that $\Sigma' \supseteq \Sigma$.*

Proof. Let $S = \{\Gamma : \Gamma \text{ is an } \mathcal{L}\text{-theory}, \Gamma \text{ is consistent, and } \Sigma \subseteq \Gamma\}$. Notice that $S \neq \emptyset$ since $\Sigma \in S$. We consider the partial order $(S, \leq)$ where $\Gamma_1 \leq \Gamma_2$ if and only if $\Gamma_1 \subseteq \Gamma_2$. We now wish to apply Zorn's lemma to this partial order. Let C be a chain in S. We need to show that C has an upper bound. Consider $\Gamma_C = \bigcup_{\Gamma \in C} \Gamma$. We claim that (1) $\Gamma_C \in S$ and (2) For any $\Gamma \in C$, $\Gamma \leq \Gamma_C$.

Claim: $\Gamma_C \in S$. It suffices to show that Γ_C is consistent. Towards a contradiction, suppose that Γ_C is inconsistent. Then $\Gamma_C \vdash (A \wedge \neg A)$ for some $A \in \mathcal{L}$. Hence there exists a proof $\theta_1, ..., \theta_n$ from Γ_C to $(A \wedge \neg A)$ where for each $i \leq n$, either

(1) θ_i is valid.
(2) $\theta_i \in \Gamma_C$.
(3) θ_i is inferred from two previous sentences in the proof.

Let $\theta_{i_1}, ..., \theta_{i_m}$ be the sentences among $\theta_1, ..., \theta_n$ which are in Γ_C. Since $\Gamma_C = \bigcup_{\Gamma \in C} \Gamma$, for each $j \leq m$, there exists $\Gamma_j \in C$ such that $\theta_{i_j} \in \Gamma_j$. Since C is a chain, the set $\{\Gamma_1, ..., \Gamma_m\}$ is totally ordered by inclusion and so we may choose $\Gamma_* \in \{\Gamma_1, ..., \Gamma_m\}$ such that for any $i \leq m$ $\Gamma_i \leq \Gamma_*$ (and so $\Gamma_i \subset \Gamma_*$). Hence for each $j \leq m$, we have that $\theta_{i_j} \in \Gamma_*$. Therefore $\theta_1, ..., \theta_n$ is a proof of $(A \wedge \neg A)$ fomr Γ_*. However $\Gamma_* \in S$ and so Γ_* is consistent. Therefore we have a contradiction.

Claim: For any $\Gamma \in C$, $\Gamma \leq \Gamma_C$. Suppose that $\Gamma \in C$. Notice that if $\varphi \in \Gamma$, then $\varphi \in \bigcup_{\Gamma \in C} \Gamma$ and so $\varphi \in \Gamma_C$. Hence $\Gamma \subseteq \Gamma_C$ and so definition $\Gamma \leq \Gamma_C$.

By Zorn's lemma, the partial order $(S, \leq)$ has a maximal element, say Γ_m. By construction, Γ_m is a maximally consistent theory which extends Σ. ☐

1.2. **Completeness and Compactness.**

Lemma 1.13. *Let* Σ *be an* $\mathcal{L}$*-theory. If* $\Sigma \vdash (\varphi_1 \to \psi)$ *and* $\Sigma \vdash (\varphi_2 \to \psi)$*, then* $\Sigma \vdash (\varphi_1 \vee \varphi_2 \to \psi)$*.*

Proof. Let $\theta_1, ..., \theta_n$ be a proof of $(\varphi_1 \to \psi)$ form Σ and let $\chi_1, ..., \chi_m$ be a proof of $(\varphi_2 \to \psi)$. We claim that the sentence $((\varphi_1 \to \psi) \to ((\varphi_2 \to \psi) \to (\varphi_1 \vee \varphi_2 \to \psi)))$ is valid (check via truth table). Let $\gamma_1 := ((\varphi_1 \to \psi) \to ((\varphi_2 \to \psi) \to (\varphi_1 \vee \varphi_2 \to \psi)))$ and $\gamma_2 := ((\varphi_2 \to \psi) \to (\varphi_1 \vee \varphi_2 \to \psi))$ We claim that

$$\theta_1, ..., \theta_n, \chi_1, ..., \chi_m, \gamma_1, \gamma_2, (\varphi_1 \vee \varphi_2 \to \psi)$$

is a proof of $(\varphi_1 \vee \varphi_2 \to \psi)$ from Σ. □

Proposition 1.14. *Suppose that* Σ *is maximally consistent.*

(1) For each φ*, either* $\varphi \in \Sigma$ *or* $\neg\varphi \in \Sigma$*.*
(2) For each pair φ, ψ*,* $\varphi \wedge \psi \in \Sigma$ *if and only if* $\varphi \in \Sigma$ *and* $\psi \in \Sigma$*.*

Proof. We prove (1). Suppose that $\varphi, \neg\varphi \notin \Sigma$. Then $\Sigma \cup \{\varphi\}, \Sigma \sup\{\neg\varphi\} \supsetneq \Sigma$. Since Σ is maximally consistent, both $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are inconsistent. Hence $\Sigma \cup \{\varphi\} \vdash (A \wedge \neg A)$ and $\Sigma \cup \{\neg\varphi\} \vdash (A \wedge \neg A)$. By the deduction theorem, we have that $\Sigma \vdash \varphi \to (A \wedge \neg A)$ and $\Sigma \vdash \neg\varphi \to (A \wedge \neg A)$. Notice that $\varphi \vee \neg\varphi$ is valid. By Lemma 1.14, $\Sigma \vdash ((\varphi \vee \neg\varphi) \to (A \wedge \neg A))$. Consider

$$\varphi \vee \neg\varphi, ((\varphi \vee \neg\varphi) \to (A \wedge \neg A)), (A \wedge \neg A).$$

We claim the above is a proof of $A \wedge \neg A$ from Σ. Hence Σ is inconsistent. □

Lemma 1.15. *Suppose that* $M \models \Sigma$*. If* $\Sigma \vdash \varphi$*, then* $M \models \varphi$*.*

Proof. This proof is by induction on the length of a proof. The Base case is left as an exercise. Induction Hypothesis: Suppose that if $\theta_1, ..., \theta_n$ is a proof of ψ from Σ, then $M \models \psi$.

Induction step: Suppose that $\theta_1, ..., \theta_{n+1}$ is a proof of φ from Σ. Consider θ_{n+1} Then one of the following is true:

(1) θ_{n+1} is valid. (Hence, θ_{n+1} is true in any model, and in particular, $M \models \theta_{n+1}$).
(2) $\theta_{n+1} \in \Sigma$. (Since $M \models \Sigma$, this implies that $M \models \theta_{n+1}$).
(3) θ_{n+1} is inferred by θ_k and θ_l where $k, l \leq n$ and $\theta_k = (\psi \to \theta_{n+1})$ and $\theta_l = \psi$.

In case (3), notice that $\theta_1, ..., \theta_k$ is a proof of θ_k from Σ and $\theta_1, ..., \theta_l$ is a proof of θ_l from Σ. By our induction hypothesis, $M \models \theta_k$ and $M \models \theta_l$. Hence $M \models \psi \to \theta_{n+1}$ and $M \models \theta_{n+1}$. We conclude that $M \models \theta_{n+1}$. □

Theorem 1.16 (Completeness Theorem)**.** Σ *is consistent if and only if* Σ *is satisfiable, i.e. there exists an* $\mathcal{L}$*-model* M *such that* $M \models \Sigma$*.*

Proof. Suppose that $M \models \Sigma$. Towards a contradiction, assume that Σ is inconsistent. Then $\Sigma \vdash (A \wedge \neg A)$ for some $A \in \mathcal{L}$. So $M \models (A \wedge \neg A)$ by Lemma 1.16. But this is a contradiction since the sentence $(A \wedge \neg A)$ is not satisfiable.

Suppose that Σ is consistent. Let $\Gamma \supseteq \Sigma$ such that Γ is maximally consistent. Let $M_\Gamma = \{A \in \mathcal{L} : A \in \Gamma\}$. Since Γ is maximally consistent, we have that $\{A \in \mathcal{L} : A \in \Gamma\} = \{A \in \mathcal{L} : \Gamma \vdash A\}$. We now argue that for any $\mathcal{L}$ sentence φ, $M_\Gamma \models \varphi$ if and only if $\Gamma \vdash \varphi$.

Base Case: Suppose that $\varphi = A$. Then $M_\Gamma \models \varphi \iff M_\Gamma \models A \iff A \in \Gamma \iff \Gamma \vdash A$.

Induction Hypothesis: Assume $M_\Gamma \models \theta \iff \Gamma \vdash \theta$ and $M_\Gamma \models \psi \iff \Gamma \vdash \psi$.

Negation: Suppose that $\varphi = \neg\psi$ and $M_\Gamma \models \varphi$. Then $M_\Gamma \models \neg\psi$ and so $M_\Gamma \not\models \psi$. By IH, $\Gamma \not\vdash \psi$. By (1) of Proposition 1.15, $\Gamma \vdash \neg\psi$. Now assume that $\Gamma \vdash \varphi$. Then $\Gamma \not\vdash \psi$ since Γ is consistent. By our induction hypothesis, $M_\Gamma \not\models \psi$. Therefore $M_\Gamma \models \neg\psi$.

Conjunction: Suppose that $\varphi = \theta \wedge \psi$. Notice that $M_\Gamma \models \theta \wedge \psi$ if and only if $M_\Gamma \models \theta$ and $M_\Gamma \models \psi$. By our induction hypothesis, this is true if and only if $\Gamma \vdash \theta$ and $\Gamma \vdash \psi$. By (2) of Proposition 1.15, this is true if and only if $\Gamma \vdash \theta \wedge \psi$

By the structural induction preformed above, $M_\Gamma \models \Gamma$ and since $\Sigma \subseteq \Gamma$, we conclude that $M_\Gamma \models \Sigma$. □

Definition 1.17. Recall that an $\mathcal{L}$-theory Σ is satisfiable if there exists a model M such that $M \models \Sigma$. We say that Σ is finitely satisfiable if for every finite subset $\Sigma_0 \subseteq \Sigma$ (i.e. $|\Sigma_0|$ is finite), Σ_0 is satisfiable (i.e. there is a model M_0 such that $M_0 \models \Sigma_0$).

Theorem 1.18 (Compactness)**.** *Σ is satisfiable if and only if Σ is finitely satisfiable.*

Proof. The forward direction is trivial. We want to prove that if Σ is finitely satisfiable, then Σ is satisfiable. Suppose not. Then Σ is not satisfiable and so by the Completeness theorem, Σ is inconsistent. Hence $\Sigma \vdash (A \wedge \neg A)$. Let $\theta_1, ..., \theta_n$ be a proof of $(A \wedge \neg A)$ from Σ. Let $\Sigma_0 := \{\theta_j : j \leq n, \theta_j \in \Sigma\}$. Notice that $|\Sigma_0|$ is finite. We claim that $\theta_1, ..., \theta_n$ is a proof of $(A \wedge \neg A)$ from Σ_0. Hence Σ_0 is inconsistent and by the completeness theorem, Σ_0 is not satisfiable. Therefore, we have shown that Σ is not finitely satisfable, a contradiction. □