PKU MODEL THEORY NOTES

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1. First-order logic

A first-order language $\mathcal{L}$ has the following symbols:

(1) Logical symbols, all languages have the following:
   (a) '(' and ')'.
   (b) Connectives, $\to, \land, \lor, \neg$.
   (c) Variables $(v_i)_{i \in \mathbb{N}}$ (In formal proofs, we have this countable of variables. In practice, we usually use the symbols x, y, z...).
   (d) An equality symbols '$=$'.
   (e) Quantifiers $\forall, \exists$.

(2) Other symbols:
   (a) A collection of function symbols (each with fixed arity). This can be possibly empty.
   (b) A collection of Relation symbols (each with fixed arity). This can be possible empty.
   (c) A collection of constant symbols. This can be possibly

In practice, we write $\mathcal{L} = \{(f_i)_{i \in I}, (R_j)_{j \in J}, (c_k)_{k \in K}\}$ where the f_i's are function symbols, the R_j's are relation symbols, and the c_k's are constant symbols.

Definition 1.1 (Atomic Formulas)**.** Let $\mathcal{L} = \{(f_i)_{i \in I}, (R_j)_{j \in J}, (c_k)_{k \in K}\}$.

(1) A term is defined as follows:
   (a) A constant symbol is a term. A variable v_i for $i \in \mathbb{N}$ is a term.
   (b) If $t_1, ..., t_n$ are terms and f is an n-ary function symbol, then $f(t_1, ..., t_n)$ is a term.

(2) An atomic formula is one of the following:
   (a) If t_1 and t_2 are terms, then $t_1 = t_2$ is an atomic formula.
   (b) If R is an n-ary relation symbol in the language, then $R(t_1, ..., t_n)$ is an atomic formula.

Definition 1.2 ($\mathcal{L}$-formula)**.** These are defined recursively:

(1) An atomic formula is a formula.
(2) If φ, ψ are formulas, then $(\neg\varphi)$, $(\varphi \land \psi)$, $(\varphi \to \psi)$, and $(\varphi \lor \psi)$ are formulas.
(3) If φ if a formula and x is a variable, then $\forall x \varphi$ and $\exists x \varphi$ are formulas.

Definition 1.3 (Free variables)**.** Let φ be a formula in a language $\mathcal{L}$ and x be a variable. We say that x occurs freely in φ if

(1) If φ is atomic, then x occurs freely in φ if and only if x appears in φ (e.g. in the atomic formula $R(x, y)$, both x and y occur freely in it).
(2) If $\varphi = (\neg\psi)$, then x occurs freely in φ if and only if x occurs freely in ψ.
(3) If $\varphi = (\psi \otimes \theta)$ where $\otimes$ is a binary connective, then x occurs freely in φ if and only if x occurs freely in ψ or (inclusively) θ.

(4) If φ is $\forall y \varphi$, then x occurs freely in φ if and only if x is free in ψ and $x \neq y$.

Definition 1.4. Let φ be an $\mathcal{L}$-formula. Then φ is said to be a sentence if no variables occur freely in φ.

2. Models and satisfaction

Definition 2.1. Let $\mathcal{L} = \{f_1, ..., f_n, R_1, ..., R_m, c_1, ..., c_k\}$. Then an $\mathcal{L}$-structure (also called an $\mathcal{L}$-model) is a tuple $(A; f_1^M, ..., R_1^M, ..., c_1^M)$ where

(1) A is a non-empty set.
(2) An interpreation for each function, relation, and constant symbol.
   (a) For each n-ary function symbol f_i in $\mathcal{L}$, $f_i^M : A^n \to A$.
   (b) For each n-ary relation symbol R_i in $\mathcal{L}$, $R_i^M \subseteq A^n$.
   (c) For each constant symbol c_i in $\mathcal{L}$, $c_i^M \in A$.

Definition 2.2 (Satisfaction)**.** Let $\mathcal{L}$ be a first-order language. Let $M = (A; ...)$ be an $\mathcal{L}$-structure. Let V be the collection of variables in $\mathcal{L}$. An assignment is a map $s : V \to A$. s extends naturally to a function $\bar{s} : T \to A$ where T is the collection of terms constructed in $\mathcal{L}$. More explicitly

(1) If $x \in V$, then $\bar{s}(x) = s(x)$.
(2) If c is a constant in $\mathcal{L}$, then $\bar{s}(c) = c^M$.
(3) If $t_1, ..., t_n$ are terms and f is an n-ary function symbol, then $\bar{s}(f(t_1, ..., t_n)) = f^M(\bar{s}(t_1), ..., \bar{s}(t_n))$.

Let φ be an $\mathcal{L}$-formula. We now define $M \models \varphi[s]$.

(1) If φ is $t_1 = t_2$; then $M \models \varphi[s]$ if and only if $\bar{s}(t_1) = \bar{s}(t_2)$.
(2) If φ is $R(t_1, ..., t_n)$, then $M \models R(t_1, ..., t_n)[s]$ if and only if $(\bar{s}(t_1), ..., \bar{s}(t_n)) \in R^M$.
(3) If φ is $(\neg\psi)$, then $M \models \varphi[s]$ if and only if $M \not\models \psi[s]$.
(4) If φ is $(\psi \wedge \theta)$, then $M \models \varphi[s]$ if and only if $M \models \psi[s]$ and $M \models \theta[s]$.
(5) If φ is $(\psi \vee \theta)$ then $M \models \varphi[s]$ if and only if $M \models \psi[s]$ or $M \models \theta[s]$.
(6) If φ is $(\psi \to \theta)$ then $M \models \varphi[s]$ if and only if $M \models (\neg\psi \vee \theta)[s]$.
(7) If φ is $\forall x \psi$, then $M \models \varphi[s]$ if and only if for every $d \in A$, $M \models \varphi[s(x|d)]$ where $s(x|d) : V \to A$, if $x \neq y$ then $s(x|d)(y) = s(y)$ and if $y = x$, then $s(y) = d$.
(8) If φ is $\exists x \psi$, then $M \models \varphi[s]$ if and only if there exists $d \in A$, $M \models \varphi[s(x|d)]$.

Proposition 2.3. *Fix $\mathcal{L}$. Let $M = (A; ...)$ be an $\mathcal{L}$-structure. Let $s_1, s_2 : V \to A$ be assignments. Let φ be an $\mathcal{L}$-formula and $F(\varphi)$ be the free variables which occur in φ. Suppose that $s_1|_{F(\varphi)} = s_2|_{F(\varphi)}$. Then $M \models \varphi[s_1]$ if and only if $M \models \varphi[s_2]$.*

Proof. Homework. □

Corollary 2.4. *Let φ be an $\mathcal{L}$-sentence and $M = (A; ...)$ be an $\mathcal{L}$-structure. Then precisely one of the following holds.*

(1) For any $s : V \to A$, $M \models \varphi[s]$.
(2) For no $s : V \to A$, $M \models \varphi[s]$.

Proof. Let $s_1, s_2 : V \to A$. Then $F(\varphi) = \emptyset$ and so $s_1|_{F(\varphi)} = s_2|_{F(\varphi)}$. By the previous proposition, $M \models \varphi[s_1]$ if and only if $M \models \varphi[s_2]$. □

Definition 2.5. If φ is an $\mathcal{L}$-sentence and M is an $\mathcal{L}$-structure. We say that $M \models \varphi$ if there exists $s : V \to A$ such that $M \models \varphi[s]$.

Definition 2.6. Let $\varphi(x_1, ..., x_n)$ be an $\mathcal{L}$-formula with free variables precisely $x_1, ..., x_n$. Let $M = (A; ...)$ be an $\mathcal{L}$-structure and $(a_1, ..., a_n) \in A^n$. We write $M \models \varphi(a_1, ..., a_n)$ and say "$(a_1, ..., a_n)$ satisfiable φ" if there exists $s : V \to A$ such that $s(x_i) = a_i$ for $1 \leq i \leq n$ and $M \models \varphi[s]$.

Definition 2.7 (Definable set)**.** . Let $M = (A; ...)$ be an $\mathcal{L}$-structure. Let $D \subseteq A^n$. We say that D is definable if there exists an $\mathcal{L}$-formula $\varphi(x_1, ..., x_n)$ such that $(a_1, ..., a_n) \in D$ if and only if $M \models \varphi(a_1, ..., a_n)$.

Definition 2.8. Let $M_1 = (A_1; ...)$ and $M_2 = (A_2; ...)$ be $\mathcal{L}$-structures. We say that M_1 is isomorphic to M_2 and write $M_1 \cong M_2$ if there exists a map $G : A_1 \to A_2$ with the following properties:

1. $G : A_1 \to A_2$ is a bijection.
2. G preserves functions, relation and constant symbols, i.e.
   - (a) For each n-ary function f_i in $\mathcal{L}$ and tuple $(a_1, ..., a_n) \in A^n$, $G(f_i^{M_1}(a_1, ..., a_n)) = f_i^{M_2}(G(a_1), ..., G(a_n))$.
   - (b) For each n-ary relation symbol R_i in $\mathcal{L}$ and tuple $(a_1, ..., a_n) \in A^n$, $(a_1, ..., a_n) \in R_i^{M_1}$ if and only if $(G(a_1), ..., G(a_n)) \in R_i^{M_2}$.
   - (c) For each constant symbol c, $G(c^{M_1}) = c^{M_2}$.

For convention, we sometimes say an isomorphism G maps from M_1 to M_2.

Definition 2.9. Let M_1 and M_2 be $\mathcal{L}$-structures. We say that M_1 is *elementary equivalent* to M_2 and write $M_1 \equiv M_2$ if for every $\mathcal{L}$-sentences φ, $M_1 \models \varphi$ if and only if $M_2 \models \varphi$.

Proposition 2.10. *Let M_1 and M_2 be $\mathcal{L}$-structures. If $M_1 \cong M_2$, then $M_1 \equiv M_2$.*

Proof. Homework. □

Example 2.11. $(\mathbb{Q}; \leq)$ and $(\mathbb{N}; \leq)$ are not elementary equivalent. $(\mathbb{R}; \leq)$ and $(\mathbb{Q}; \leq)$ are not isomorphic, but they are elementary equivalent.

Example 2.12. Consider the language $\mathcal{L} = \{f\}$ where f is a unary function symbol. Consider the structure $M = (\mathbb{N}, f^M)$ where $f^M = S$ is the usual successor function. Let $\mathbb{E} = \{n \in \mathbb{N} : n \text{ is even}\}$ and $S' : \mathbb{E} \to \mathbb{E}$ via $S'(n) = n + 2$. Let $N = (\mathbb{E}, f^N)$ where $f^N = S'$. Consider the map $G : M \to N$ via $G(n) = n + 2$. We claim that this map is an isomorphism.