PKU MODEL THEORY NOTES

Exercise 0.1. *Let $M = (G; R)$ be a simple graph, i.e. $R(x, y)$ is a binary relation such that*

(1) $M \models \forall x \neg R(x, x)$.
(2) $M \models \forall x \forall y (R(x, y) \to R(y, x))$.

Write down the following formulas:

(1) $M \models \varphi(a)$ *iff the degree of a is* 4.
(2) $M \models \varphi(a, b, c)$ *iff the induced graph on $\{a, b, c\}$ forms a triangle.*
(3) $M \models \varphi(a)$ *iff one of a's neighbors has degree* 3.
(4) $M \models \varphi_n(a, b)$ *iff the shortest path from a to b is of length n.*
(5) $M \models \varphi_n(a, b, c)$ *iff there exists a path of from a to b avoiding c of length less than n.*

Example 0.2. $(\mathbb{R}; \leq)$ and $(\mathbb{Q}; \leq)$ are not isomorphic, but they are elementary equivalent.

Definition 0.3. Fix $\mathcal{L}$ and let $M = (A; ...)$ be an $\mathcal{L}$-structure. A map $G : A \to A$ is called an automorphism if it is an isomorphism.

Proposition 0.4. *Let $M = (A; ...)$ and $G : M \to M$ be an automorphism. Let $D \subseteq A^n$ be a definable set. Then for any $(a_1, ..., a_n) \in A^n$, we have that $(a_1, ..., a_n) \in D$ if and only if $(G(a_1), ..., G(a_n)) \in D$.*

Proof. We prove this statement via induction on the complexity of a formula. We will show that if $(a_1, ..., a_n) \in D$ then $(G(a_1), ..., G(a_n)) \in D$. Noticing that G^{-1} is also an automorphism finishes the claim. Suppose that $(a_1, ..., a_n) \in D$ and $\varphi(x_1, ..., x_n)$ is a definition of D.

Claim: For any term t and $s : V \to A$, we have that $G(\bar{s}(t)) = \overline{(G \circ s)}(t)$.

(1) If x is a variable, then $G(\bar{s}(x)) = G(s(x)) = (G \circ s)(x) = \overline{(G \circ s)}(s)$.
(2) If c is a constant, then $G(\bar{s}(c)) = G(s(c)) = (G \circ s)(c) = \overline{(G \circ s)}(c)$.
(3) Assume that $t_1, ..., t_n$ are terms such that $G(\bar{s})(t_i) = \overline{(G \circ s)}(t_i)$ for $i \leq n$. Let f be an n-ary function symbol. Then $G(\bar{s}(f(t_1, ..., t_n)) = G(f^M(\bar{s}(t_1), ..., \bar{s}(t_n))$
$= f^M(G(\bar{s}(t_1)), ..., G(\bar{s}(t_n))) = f^M(\overline{(G \circ s)}(t_1), ..., \overline{(G \circ s)}(t_n))) = \overline{(G \circ s)}(f(t_1, ..., t_n))$.

We now show the claim via induction.

(1) Suppose that $\varphi(x_1, ..., x_n)$ is atomic. Then $\varphi(x_1, ..., x_n)$ is $R(t_1, ..., t_m)$ where R is a relation symbol in $\mathcal{L}$ and $t_1, ..., t_m$ are terms. Suppose that

$(a_1, ..., a_n) \in D$. Let $s : V \to A$ where $s(x_i) = a_1$. Then

$$\begin{aligned}(a_1, ..., a_n) \in D &\iff M \models \varphi[s] \\ &\iff (\bar{s}(t_1), ..., \bar{s}(t_m)) \in R^M \\ &\iff G(\bar{s}(t_1)), ..., G(\bar{s}(t_m))) \in R^M \\ &\iff (\overline{(G \circ s)}(t_1), ..., \overline{(G \circ s)}(t_m)) \in R^M \\ &\iff M \models \varphi[G \circ s] \\ &\iff (G(a_1), ..., G(a_n)) \in D.\end{aligned}$$

(2) Induction Hypothesis: Suppose that ψ and θ are formulas. Assume that if $M \models \psi[s]$ if and only if $M \models \psi[G \circ s]$ and $M \models \theta[s]$ if and only if $M \models \psi[G \circ s]$. We notice that we implicitly proved this statement in the first step. Moreover, we implicitly prove this hypothesis in the forthcoming steps as well.

(3) Suppose that $\varphi(x_1, ..., x_n)$ is $\neg\psi(x_1, ..., x_n)$. Let $(a_1, ..., a_n) \in D$ and $s : V \to A$ such that $s(x_i) = a_i$ and $(a_1, ..., a_n) \in D$. Then

$$\begin{aligned}(a_1, ..., a_n) \in D &\iff M \models \varphi[s] \\ &\iff M \models \neg\psi[s] \\ &\iff M \not\models \psi[s] \\ &\iff M \not\models \psi[G \circ s] \\ &\iff M \models \neg\psi[G \circ s] \\ &\iff M \models \varphi[G \circ s] \\ &\iff (G(a_1), ..., G(a_n)) \in D.\end{aligned}$$

(4) Suppose that $\varphi(x_1, ..., x_n)$ is $(\psi \wedge \theta)(x_1, ..., x_n)$. Let $(a_1, ..., a_n) \in D$ and $s : V \to A$ where $s(x_i) = a_i$. Then

$$\begin{aligned}(a_1, ..., a_n) \in D &\iff M \models \varphi[s] \\ &\iff M \models (\psi \wedge \theta)[s] \\ &\iff M \models \psi[s] \text{ and } M \models \theta[s] \\ &\iff M \models \psi[G \circ s] \text{ and } M \models \theta[G \circ s] \\ &\iff M \models (\psi \wedge \theta)[G \circ s] \\ &\iff M \models (G(a_1), ..., G(a_n)) \in D.\end{aligned}$$

(5) Suppose that $\varphi(x_1, ..., x_n)$ is $\exists x\psi(x_1, ..., x_n)$. Let $(a_1, ..., a_n) \in D$ and $s : V \to A$ where $s(x_i) = a_i$. Then

$$\begin{aligned}(a_1, ..., a_n) \in D &\iff M \models \exists x\psi[s] \\ &\iff \text{There exists some } d \text{ in } A \text{ such that } M \models \psi[s(x|d)] \\ &\iff M \models \psi[G \circ s(x|d)] \\ &\iff M \models \psi[(G \circ s)(x|G(d))] \\ &\iff M \models \exists x\psi[G \circ s] \\ &\iff (G(a_1), ..., G(a_n)) \in D.\end{aligned}$$

□

The previous result allows us to show that certain subsets of a first order structures are not definable.

Example 0.5. Consider $(\mathbb{Z}, S)$ where S is the usual successor function. Let $\mathbb{E} = \{n \in \mathbb{Z} : n \text{ is even}\}$. Consider the map $\sigma : \mathbb{Z} \to \mathbb{Z}$ via $\sigma(n) = n + 1$. We claim that σ is an automorphism. Notice that if $\mathbb{E}$ where definable, then for any $a \in \mathbb{E}$, we have that $\sigma(a) \in \mathbb{E}$. However, $2 \in \mathbb{E}$, $\sigma(2) = 3$, and $3 \notin \mathbb{E}$. Hence $\mathbb{E}$ is not definable in this structure.

Definition 0.6. A collection of $\mathcal{L}$-sentences is called a an $\mathcal{L}$-theory, or sometimes just a theory.

Definition 0.7. A collection of $\mathcal{L}$-structures $\mathcal{K}$ is called an elementary class if there exists an $\mathcal{L}$-theory Σ such that $M \in \mathcal{K}$ if and only if $M \models \Sigma$.

Example 0.8. Let $\mathcal{L} = \{\leq\}$. Then linear orders form an $\mathcal{L}$-elementary class. Consider the following sentences:

(1) $\varphi_1 = \forall x(x \leq x)$.
(2) $\varphi_2 = \forall x \forall y \forall z((x \leq y \wedge y \leq z) \to x \leq z)$.
(3) $\varphi_3 = \forall x \forall y((x \leq y \wedge y \leq x) \to x = y)$.
(4) $\varphi_4 = \forall x(x \leq y \vee y \leq x)$

Let $\Sigma = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$. Then Σ shows that linear orders form an elementary class.

Example 0.9. Let $\mathcal{L} = \{\emptyset\}$. Then the class of infinite $\mathcal{L}$-structures form an elementary class. For each $n \in \mathbb{N}\backslash\{0\}$, consider the sentence

$$\varphi_n = \exists x_1 ... \exists x_n \left(\bigwedge_{1 \leq i,j \leq n; i \neq j} x_i \neq x_j \right)$$

Then $\Sigma = \{\varphi_n : n \geq 1\}$ witnesses the fact that the class of infinite $\mathcal{L}$-structures is an elementary class.

1. Compactness theorem

Theorem 1.1 (Compactness theorem)**.** *Let Σ be an $\mathcal{L}$-theory. Then Σ is satisfiable if and only if Σ is finitely satisfiable. In other words, there exists some M such that $M \models \Sigma$ if and only if for every $\Sigma_0 \subset \Sigma$ such that Σ_0 is finite, there exists some $M_0 \models \Sigma_0$.*

Proposition 1.2. *Let $\mathcal{L} = \{\emptyset\}$. Then the class of finite structures is not an $\mathcal{L}$-elementary class.*

Proof. Suppose that the class of finite structures is an $\mathcal{L}$-elementary class. Let Σ witness this property. Consider $\mathcal{L}' = \mathcal{L} \cup \{c_i : i \in \mathbb{N}\}$. Consider $\Sigma' = \Sigma \cup \{c_i \neq c_j : (i,j) \in \mathbb{N}^2, i \neq j\}$. We claim that Σ' is finitely satisfiable (check). Hence Σ' is satisfiable and so there exists some $\mathcal{L}'$-structures $M = (A; (c_i^M)_{i \in \mathbb{N}})$ such that $M \models \Sigma'$. Notice that by construction, we have that $|A|$ is infinite. Consider $M_* = (A;)$. Then M_* is an $\mathcal{L}$-structure and $M_* \models \Sigma$, but $|A|$ is infinite. This is a contradiction. □

Definition 1.3. Let $(I; \leq)$ be a linear order. We say that $(I; \leq)$ is well ordered if for any subset S of I, S has a least element.

Example 1.4. $(\mathbb{N}, \leq)$ is well-ordered, $(\mathbb{Q}, \leq)$ is not well-ordered.

Proposition 1.5. *Let $\mathcal{L} = \{\leq\}$. Then the class of well-ordered linear orders is not an elementary class.*

Proof. Suppose that the class of well ordered linear orders forms an $\mathcal{L}$-elementary class and let Σ be the collection of $\mathcal{L}$ which witnesses this property. Consider $\mathcal{L}' = \mathcal{L} \cup \{c_i : i \in \mathbb{N}\}$. Let $\Sigma' = \Sigma \cup \{c_i > c_{i+1} : i \in \mathbb{N}\}$. We claim that Σ' is finitely satisfiable (check). Hence there exists an $\mathcal{L}'$-structure $M' = (A, \leq^{M'}, (c_i^{M'})_{i \in \mathbb{N}})$ such that $M' \models \Sigma'$. Notice that the set $S = \{a \in A : \exists n, c_n^{M'} = a\}$ is not well-ordered. Consider $M_* = (A, \leq^{M'})$. Then $M_* \models \Sigma$ and M_* is not well-ordered (since S is still a subset of A). Hence, we have a contradiction. $\square$

Example 1.6. Let T be an $\mathcal{L}$-theory. If T has arbitrarily large finite models, then T has an infinite model

Proof. The following idea is a good one to keep in mind.

(1) Expand language $\mathcal{L}$ to $\mathcal{L}'$ where $\mathcal{L}' = \mathcal{L} \cup C$ where C is a collection of *new* constant symbols. Consider a $\mathcal{L}'$-theory T' such that $T \subseteq T'$.
(2) Aim to apply compactness by turning $\mathcal{L}$-structures into $\mathcal{L}$'-structures.
(3) Show that T' is finitely satisfiable using the structures constructed in the previous step. Apply the compactness theorem to get a $\mathcal{L}$'-structure N. Note $N \models T'$.
(4) Forget about the interpretation of the constants to get an $\mathcal{L}$-structure N_0 such that $N_0 \models T$.

Let $C = \{c_i : i \in \mathbb{N}\}$ be a collection of new constant symbols which do not appear in $\mathcal{L}$. We let $\mathcal{L}' = \mathcal{L} \cup C$. Consider the theory $T' = T \cup \{c_i \neq c_j : i \neq j\}$. We claim that T is finitely consistent. Indeed, let Δ be a finite subcollection of T'. Let $k = \max\{n : c_n$ appears in one of the sentences in $\Delta\}$. Then $\Delta \subseteq T \cup \{c_i \neq c_j : 0 \leq i \neq j \leq k\}$. Since T has arbitrarily large finite models, there exists some $\mathcal{L}$-structure M_0 such that $M_0 \models T$ and $|M_0| \geq k$. We can turn M_0 into a $\mathcal{L}'$-structure by giving interpretations for each constant symbol. Since $|M_0| \geq k$, we can choose an injection $f : \{c_1, ..., c_k\} \to M_0$ and let M_0' be the $\mathcal{L}'$-structure given setting $c_i^{M_0'} = f(c_i)$. Now $M_0' \models \Delta$.

By the compactness theorem, there exists some $\mathcal{L}'$-structure N such that $N \models T'$. Notice that we have an injection from $\mathbb{N} \to N$ via $f(i) = c_i^N$. Hence $|N| \geq \aleph_0$. Moreover, since $T \subseteq T'$, $N \models T'$. Technically N is an $\mathcal{L}'$ structure. To get an $\mathcal{L}$-structure, one should simply forget about the interpretation of the new constant symbols. $\square$