PKU MODEL THEORY NOTES

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Definition 0.1. We say that an $\mathcal{L}$-sentence φ is valid if it is true in every $\mathcal{L}$-structure.

Example 0.2. Check that the following are valid:

(1) $\forall x(x = x)$.
(2) Let $\varphi(x)$ be an $\mathcal{L}$-formula with one free variable x. Then $\forall x\varphi(x) \to \forall y\varphi(x|y)$ is valid where $\varphi(x|y)$ is the formula where each free occurrence of x is replaced by a new variable y which does not appear in $\varphi(x)$.

Definition 0.3 (Model Theorist's Proof System)**.** Let Σ be an $\mathcal{L}$-theory. Then $\Sigma \vdash \varphi$ if and only if there exists $\theta_1, ..., \theta_n$ a finite sequence of $\mathcal{L}$-sentences such that $\theta_n = \varphi$ and for each $i \leq n$, either

(1) θ_i is valid.
(2) $\theta_i \in \Sigma$.
(3) (Modus Ponens) There exists $k, j < i$ such that $\theta_j = \psi \to \theta_i$ and $\theta_k = \psi$.

Definition 0.4. We say that Σ is consistent if there exists some $\mathcal{L}$-sentence φ such that $\Sigma \not\vdash \varphi$.

Proposition 0.5. *The following are equivalent.*

(1) Σ *is inconsistent.*
(2) For every $\mathcal{L}$*-sentence* φ*,* $\Sigma \vdash \varphi \wedge \neg\varphi$*.*
(3) There exists an $\mathcal{L}$*-sentence* φ *such that* $\Sigma \vdash \varphi \wedge \neg\varphi$*.*

Proof. Similar to propositional logic. □

1. Basics of Proofs

Definition 1.1. We say that Σ is maximally consistent if Σ is consistent and for any $\Sigma' \supsetneq \Sigma$, Σ' is inconsistent.

Lemma 1.2. *Assume that* $\Gamma \vdash \varphi$ *and* c *is a constant symbol which occurs in no sentence in* Γ*. Suppose that* y *is a variable which does not occur in* φ*. Then* $\Gamma \vdash \forall y\varphi(c|y)$ *where* $\varphi(c|y)$ *is the formula constructed by replacing every occurrence of the constant* c *with the variable* y*. Moreover, there is a proof of* $\forall y\varphi(c|y)$ *from* Γ *where* c *does not occur.*

Proof Sketch. Let $\theta_1, ..., \theta_n$ be a proof of φ from Γ. Let y be a variable which does not occur in any of the θ_i's. We claim that $\forall y\theta_1(c|y), ..., \forall y\theta_n(c|y)$ is a proof of $\forall y\varphi(c|y)$. □

Lemma 1.3. *[Deduction Theorem] If* $\Gamma \cup \{\psi\} \vdash \varphi$*, then* $\Gamma \vdash \psi \to \varphi$*.*

Lemma 1.4. *If* Γ *is consistent and* $\Gamma \cup \{\psi\}$ *is inconsistent, then* $\Gamma \vdash \neg\psi$*.*

Proof. Assume the above. Since $\Gamma \cup \{\psi\}$ is inconsistent, $\Gamma \cup \{\psi\} \vdash \varphi \wedge \neg\varphi$ for some/any $\mathcal{L}$-sentence φ. By the deduction theorem, $\Gamma \vdash \psi \to \varphi \wedge \neg\varphi$. Let $\theta_1, ..., \theta_n$ be a proof of $\psi \to \varphi \wedge \neg\varphi$ from Γ. Claim: $(\psi \to \varphi \wedge \neg\varphi) \to \neg\psi$ is valid. So, $\theta_1, ..., \theta_n, (\psi \to \varphi \wedge \neg\varphi), \neg\psi$ is a proof of $\neg\psi$ from Γ. □

Lemma 1.5 (Lindenbaum's Theorem)**.** *If Σ is a consistent $\mathcal{L}$-theory, then $\exists\Gamma$ such that $\Gamma \supseteq \Sigma$ and Γ is maximally consistent.*

Proof. Zorn's lemma, almost identical to propositional logic. □

2. Building Models

Definition 2.1. Let Σ be a set of $\mathcal{L}$-sentences. Let C be a subset of the constant symbols in $\mathcal{L}$. We say that C is a set of witnesses for Σ in $\mathcal{L}$ if and only if for every formula φ with at most one free variable, say x, there exists a constant symbol $c \in C$ such that

$$\Sigma \vdash \exists x\varphi \to \varphi(x|c)$$

where $\varphi(x|c)$ is the sentence where each free instance of x is replaced by c.

Lemma 2.2. *Let Σ be a consistent set of $\mathcal{L}$-sentences. Let C be a set of new constant symbols such that $|C| = |\mathbb{N}|$ if $|\mathcal{L}|$ is finite or countable and $|C| = |\mathcal{L}|$ otherwise. Let $\overline{L} = \mathcal{L} \cup C$. Then there exists $\overline{\Sigma}$, an $\overline{\mathcal{L}}$-theory such that*

(1) $\overline{\Sigma} \supset \Sigma$.
(2) $\overline{\Sigma}$ is consistent.
(3) $\overline{\Sigma}$ has witnesses in $\overline{\mathcal{L}}$, namely C.

Proof. We prove the countable case. Suppose that $|\mathcal{L}| \leq |\mathbb{N}|$. Let $C = \{k_i : i \in \mathbb{N}\}$. Check that $|\{\varphi : \varphi \text{ is an } \overline{\mathcal{L}}\text{-formula }\}| = |\mathbb{N}|$. Hence $|\{\varphi | \varphi \text{ is an } \overline{\mathcal{L}}\text{-formula with at most one free variable}\}| = |\mathbb{N}|$. Enumerate this set, $\varphi_1, \varphi_2, ...$ We now construct $\overline{\Sigma}$.

(1) Step 1: Let $\Sigma_1 = \Sigma$.
(2) Step $n+1$: Suppose that we have constructed Σ_n. We let

$$\Sigma_{n+1} = \Sigma_n \cup \{\exists x_i \varphi_i \to \varphi_i(x_i|d_i)\}$$

where x_i is the free variable which occurs in φ_i (if no free variable occurs, we treat x_i simply as v_0). We let d_i be the first constant symbol in $k_0, k_1, k_2, ...$ which does not occur in Σ_n.
(3) Let $\overline{\Sigma} = \bigcup_{i \in \mathbb{N}} \Sigma_i$.

Notice that

(1) $\Sigma \subseteq \overline{\Sigma}$.
(2) By construction, $\overline{\Sigma}$ has a set of witnesses in $\overline{L}$.
(3) We need to check that $\overline{\Sigma}$ is consistent. Suppose not. Since proofs are finitary, there exists some j such that Σ_j is inconsistent. Let j be the small Σ_j such that Σ_j is inconsistent. So, Σ_{j-1} is consistent and $\Sigma_j = \Sigma_{j-1} \cup \{\exists x_j \varphi_j \to \varphi_j(x_j|d_j)\}$ is inconsistent. By Lemma 1.4, we have that

$$\Sigma_{j-1} \vdash \neg(\exists x_j \varphi_j \to \varphi_j(x_j|d_j))$$

and so one can check,

$$\Sigma_{j-1} \vdash \exists x_j \varphi_j \wedge \neg\varphi_j(x_j|d_j))$$

and so by Lemma 1.2,

$$\Sigma_{j-1} \vdash \forall y(\exists x_j \varphi_j \wedge \neg\varphi_j(y))$$

and so one can check,

$$\Sigma_{j-1} \vdash \exists x_j \varphi_j \wedge \forall y(\neg\varphi_j(y))$$

and so one can check,

$$\Sigma_{j-1} \vdash \exists x_j \varphi_j \wedge \neg\exists y \varphi_j(y))$$

and so one can check,

$$\Sigma_{j-1} \vdash \exists x_j \varphi_j \wedge \neg\exists x_j \varphi_j(y|x_j))$$

Hence Σ_{j-1} is inconsistent, a contradiction.

Therefore, $\overline{\Sigma}$ has the appropriate properties. □

Definition 2.3 (Equivalence relations)**.** Suppose that X is a set. An equivalence relation $\sim$ on X is a relation such that

(1) For every $x \in X$, $x \sim x$.
(2) For every $x, y \in X$, if $x \sim y$, then $y \sim x$.
(3) For every $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

If X is a set and $\sim$ is an equivalence relation on X, we let $\tilde{x} = \{y \in X : x \sim y\}$. $\tilde{x}$ is called the equivalence class of x. Finally, $X/\sim = \{\tilde{x} : x \in X\}$.

Lemma 2.4. *Let Σ be a consistent set of $\mathcal{L}$-sentences. Suppose that C is a set of witnesses for Σ in $\mathcal{L}$. Then there exists M such that $M \models \Sigma$.*

Proof. By Lindenbaum's theorem, there exists Γ such that $\Sigma \subseteq \Gamma$ and Γ is maximally consistent. Note that C is a set of witnesses for Γ in $\mathcal{L}$. We define an equivalence relation $\sim$ on C as follows; For any $a, b \in C$, we say that $a \sim b$ if and only if $a = b \in \Gamma$. We let $A = \{\tilde{c} : c \in C\}$. We now build a model of Γ. We let $M = (A; ...)$. We now need to give interpretations to relation symbols, constant symbols, and function symbols.

(1) Suppose that R is an n-ary relation symbol in $\mathcal{L}$. We let $(\tilde{c}_1, ..., \tilde{c}_n) \in R^M$ if and only if $R(c_1, ..., c_n) \in \Gamma$. This is well-defined since $R(c_1, ..., c_n) \wedge \bigwedge_{i=1}^n c_i = d_i \rightarrow R(d_1, ..., d_n)$ is valid.
(2) Suppose that e is a constant symbol in $\mathcal{L}$. Then $\exists v_0(e = v_0)$ is valid and so $\Gamma, \Sigma \vdash \exists v_0(e = v_0)$. Then '$e = v_0$' is a formula with one free variable. Since C is a set of witnesses for Γ, there exists some $c \in C$ such that
$$\Gamma, \Sigma \vdash \exists v_0(e = v_0) \rightarrow e = c$$
Hence $e = c \in \Gamma$ for some $c \in C$. We let $e^M = \tilde{c}$ and claim that this is well-defined.
(3) Suppose that f is an n-ary function symbol in $\mathcal{L}$. Let $c_1, ..., c_n \in C$. Then $\Gamma \vdash \exists v_0(f(c_1, ..., c_n) = v_0)$ and since C is a set of witnesses for Γ, $\Gamma \vdash f(c_1, ..., c_n) = c$ for some $c \in C$. We let $f^M(\tilde{c}_1, ..., \tilde{c}_n) = \tilde{c}_m$ if and only if $f(c_1, ..., c_n) = c_m \in \Gamma$ and claim that this is also well-defined.

We now argue that $M \models \Gamma$. Everything is more or less straightforward. The base case follows via construction and $\wedge, \neg$ are as usual *downhill* proofs. We prove the case with quantifiers via induction. We let $Q(\psi)$ be the number of quantifiers in ψ. We suppose that if $Q(\psi) < n$, then $M \models \psi$ if and only if $\psi \in \Gamma$. Let $\varphi = \exists x\psi$ and $Q(\varphi) = n$.

Suppose that $M \models \varphi$. Then $M \models \exists x\psi$. So, there exists some $\tilde{c} \in A$ such that $M \models \varphi[s]$ where $s(x) = \tilde{c}$. Hence $M \models \psi(x|c)$ where $\psi(x|c)$ is obtained by replacing each free occurrence of x with the constant symbol c. By our induction hypothesis, $\psi(x|c) \in \Gamma$. Since $\psi(x|c) \to \exists x\psi$ is valid, we conclude that $\Gamma \vdash \exists x\psi$ and since Γ is maximally consistent, $\exists x\psi \in \Gamma$.

Suppose that $\varphi \in \Gamma$. Recall that Γ has witnesses in C. Hence $\Gamma \vdash \exists x\psi \to \psi(x|c)$ for some $c \in C$. Since Γ is maximal, $\Gamma \vdash \psi(x|c)$. By our induction hypothesis, $M \models \psi(x|c)$ and hence $M \models \exists x\psi$.

We conclude that the structure M is a model of Γ. □

Theorem 2.5 (Completeness Theorem)**.** *Let Σ be an $\mathcal{L}$-theory. Σ is consistent if and only if Σ is satisfiable.*

Proof. Satisfiable ($\Rightarrow$) consistent since "Models respect deductions". We show that consistent ($\Rightarrow$) satisfiable. Consider $\overline{\mathcal{L}} = \mathcal{L} \cup C$ where $|C|$ is countable if $|\mathcal{L}|$ is finite or countable, otherwise, we take $|C| = |\mathcal{L}|$. Let $\overline{\Sigma}$ be an $\overline{\mathcal{L}}$-theory such that

1. $\overline{\Sigma} \supset \Sigma$.
2. $\overline{\Sigma}$ is conisistent.
3. $\overline{\Sigma}$ has a set of witnesses in C.

Then $\exists M$, an $\overline{\mathcal{L}}$-structure such that $M \models \overline{\Sigma}$. Let M_* be the $\mathcal{L}$-structure obtained by forgetting about the new constant symbols. Then $M_* \models \Sigma$. □

Theorem 2.6 (Compactness Theorem)**.** *Σ is satisfiable if and only if Σ is finitely satisfiable. In other words, $\exists M$ such that $M \models \Sigma$ if and only if for any $\Sigma_0 \subseteq_{finite} \Sigma$, there exists M_0 such that $M_0 \models \Sigma_0$.*

Proof. Homework/Same as the proof as in propositional logic (from the completeness theorem). □

3. Categoricity

Definition 3.1. Let X be a set. We say that X is countable if there exists a bijection $f : X \to \mathbb{N}$. We also write $|X| = \aleph_0$ to mean this. If κ is any cardinal, we say that X has size κ if there exists a bijection $f : X \to \kappa$ (and again, write $|X| = \kappa$).

Definition 3.2. Let Σ be an $\mathcal{L}$-theory. We say that Σ is κ-categorical if for any $M_1, M_2 \models \Sigma$, $|M_1| = |M_2| = \kappa \implies M_1 \cong M_2$. We say that Σ is countably categorical if Σ is $\aleph_0$-categorical.

Definition 3.3. We say that Σ is complete if

1. Σ is consistent.
2. For any $\mathcal{L}$-sentence φ, either $\Sigma \vdash \varphi$ (exclusively) or $\Sigma \vdash \neg\varphi$

Example 3.4. Let M be an $\mathcal{L}$-structure. Then $Th_{\mathcal{L}}(M) := \{\varphi : M \models \varphi\}$ is a complete $\mathcal{L}$-theory.