PKU MODEL THEORY NOTES

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1. Basic set theory

Formally, all the the mathematics we are doing is inside a model of ZFC (which we call V). V is a structure in the language $\mathcal{L} = \{\in\}$.

Definition 1.1. An ordinal is an element α in V such that $(\alpha, \in)$ is a well-order and α is transitive, i.e. if $\beta \in \alpha$ and $\gamma \in \beta$, then $\gamma \in \alpha$.

Fact 1.2. *The class of ordinals is well-ordered.*

Example 1.3. Here are some examples of ordinals:

(1) $(\emptyset, \in)$ - We write this as 0.
(2) $(\{\emptyset\}, \in)$ - We write this as 1.
(3) $(\{\{\emptyset\}, \emptyset\}, \in)$ - We write this as 2.
(4) $(\{\{\{\emptyset\}, \emptyset\}, \{\emptyset\}, \emptyset\}, \in)$ - We write this as 3.
(5) For a finite number $n+1$, we have that $(n+1, \in) = (\{n\} \cup n, \in)$.
(6) $\omega = \bigcup_{n \in \mathbb{N}} n$. ω is the first infinite ordinal and has order type $(\mathbb{N}, \leq)$.
(7) $(\omega + 1, \in) = (\{\omega\} \cup \omega, \in)$
(8) One can keep going and going... $\omega + \omega$, ω^2,...,
(9) ω_1 is the first uncountable ordinal...
(10) ω_α is the α-th uncountable ordinal such that there is no bijection from ω_α to any previous ordinal.

Example 1.4. We try to draw some pictures in latex:

(1) 1 looks like $*$.
(2) 5 looks like $*****$.
(3) ω looks like the natural numbers, $****....$ We will also write ω as $\rightarrow$ in the following pictures.
(4) $\omega + 1$ looks like $\rightarrow *$. It is the order type of the set $\{1 - \frac{1}{n} : n \geq 1\} \cup \{2\}$.
(5) $\omega + 2$ looks liks $\rightarrow **$.
(6) $\omega + \omega$ looks like $\rightarrow\rightarrow$. It is the order type of the set $\{1 - \frac{1}{n} : n \geq 1\} \cup \{5 - \frac{1}{n : n \geq 1}\}$.
(7) ω^2 looks like $\underbrace{\omega + \omega + \omega + ...}_{\omega\text{-many times}}$. So it looks like $\rightarrow\rightarrow\rightarrow\rightarrow ....$ Maybe another way to think about it is to look at ω and replace each $*$ with $\rightarrow$.
(8) Now you can consider $\omega^2 + \omega + 1$.
(9) Notice that $\omega^2 + 1 + \omega$ has the same order type as $\omega^2 + \omega$.

Remark 1.5. If A is a set of ordinals. Then $\bigcup A$ is an ordinal.

Definition 1.6. A cardinal is an ordinal such that there is no bijection with any previous ordinal. In other words, κ is a cardinal if for any $\alpha \in \kappa$, there is not bijection from κ to α.

Definition 1.7. We have the following canonical notation:

(1) We let $\aleph_0$ denote ω, or the size of the natural numbers. Formally, there is no difference between $\aleph_0$ and ω. We just treat them differently.
(2) For any ordinal α, we let $\aleph_\alpha$ denote the α-th uncountable ordinal. Formally, there is no difference between ω_α and $\aleph_\alpha$. However, we treat them differently. ω_α emphasizes the order type while $\aleph_\alpha$ emphasizes the cardinality.

Proposition 1.8. *For every set A, $|A| < \mathcal{P}(A)$*

Proof. Suppose that there is a bijection $f : A \to \mathcal{P}(A)$. Let $K = \{a \in A : a \notin f(a)\}$. Since f is a bijection, there exists some $b \in A$ such that $f(b) = K$. Now we ask the question: is $b \in K$ or is $b \notin K$? Suppose that $b \in K$. Then $b \notin f(b)$. Then $b \notin K$. Okay, so this cannot happen. What if $b \notin K$. Then $b \notin f(b)$. Then $b \in K$, but this is also a problem. So we have a contradiction. There is no bijection. Our proof actually shows that there is no surjection. □

2. Categoricity

Definition 2.1. Let X be a set. We say that X is countable if there exists a bijection $f : X \to \mathbb{N}$. We also write $|X| = \aleph_0$ to mean this. If κ is any cardinal, we say that X has size κ if there exists a bijection $f : X \to \kappa$ (and again, write $|X| = \kappa$).

Definition 2.2. Let Σ be an $\mathcal{L}$-theory. We say that Σ is κ-categorical if for any $M_1, M_2 \models \Sigma$, $|M_1| = |M_2| = \kappa \implies M_1 \cong M_2$. We say that Σ is countably categorical if Σ is $\aleph_0$-categorical.

Definition 2.3. We say that Σ is complete if

(1) Σ is consistent.
(2) For any $\mathcal{L}$-sentence φ, either $\Sigma \vdash \varphi$ (exclusively) or $\Sigma \vdash \neg\varphi$

Example 2.4. Let M be an $\mathcal{L}$-structure. Then $Th_{\mathcal{L}}(M) := \{\varphi : M \models \varphi\}$ is a complete $\mathcal{L}$-theory.

Theorem 2.5 (Vaught's Test)**.** *Suppose that $|\mathcal{L}| \leq \aleph_0$, Σ has no finite models, and Σ is countably categorical. Then Σ is complete.*

Proof. Suppose that Σ is incomplete. Then there exists an $\mathcal{L}$-sentence φ such that $\Sigma \nvdash \varphi$ and $\Sigma \nvdash \neg\varphi$. We claim that $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are consistent. By our proof of the completeness theorem, there exists M_1 and M_2 such that $M_1 \models \Sigma \cup \{\varphi\}$, $M_2 \models \Sigma \cup \{\neg\varphi\}$, and $|M_1| = |M_2| = \aleph_0$. Then $M_1 \not\equiv M_2$ and so $M_1 \ncong M_2$, a contradiction. □

Theorem 2.6 (Extended Vaught's Test)**.** *Suppose that $|\mathcal{L}| \leq \kappa$, Σ has no finite models, and Σ is λ-categorical for some $\lambda \geq \kappa$. Then Σ is complete.*

Proof. Same as previous theorem. □

Example 2.7. Let $\mathcal{L} = \emptyset$. Consider $T = \{\exists x_1 ... \exists x_n \left(\bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j \right) | n \geq 1\}$. Then T is countable categorical and complete.

Proof. Let $M_1 = (A;)$ and $M_2 = (B;)$ be countable models of T. Since $|M_1| = \aleph_0$, there exists a bijection $f : A \to \mathbb{N}$. Since $|M_2| = \aleph_0$, there exists a bijection $g : B \to \mathbb{N}$. Notice that $G := g^{-1} \circ f : A \to B$ is a bijection which preserves relations, functions and constant symbols (since there are none). Hence G is an isomorphism and $M_1 \cong M_2$. Therefore, T is countable categorical. T does not have any finite models and so by Vaught's test, T is complete. $\square$

Example 2.8. Consider $\mathcal{L} = \{\leq\}$. We let $T_{\leq}$ be the theory consisting of the following sentences:

(1) $\forall x(x \leq x)$.
(2) $\forall x \forall y(x \leq y \wedge y \leq x \to x = y)$.
(3) $\forall x \forall y(x \leq y \vee y \leq x)$
(4) $\forall x \forall y \forall z(x \leq y \wedge y \leq z \to x \leq z)$.
(5) $\forall x \forall y \exists z(x \leq y \wedge x \neq y \to x < y < z)$ (here $<$ is an abbreviation).
(6) $\forall x \exists y(x \leq y \wedge x \neq y)$.
(7) $\forall x \exists y(y \leq x \wedge x \neq y)$.

The theory above is known as *Dense linear orders without endpoints* and is sometimes abbreviated as *DLO*. This theory is countable categorical and complete.

Proof. We know that $(\mathbb{Q}, \leq)$ is a countable model of $T_{\leq}$. We let $N \models T_{\leq}$ such that $|N| = \aleph_0$. We now construct an isomorphism between them.

(1) Enumerate $\mathbb{Q}$: $a_1, a_2, a_3, ...$
(2) Enumerate $N : b_1, b_2, b_3, ...$
(3) We construction an isomorphism in stages. Step one, Let $dom(f_1) = \{a_1\}$ and set $f_1(a_1) = b_1$.
(4) Suppose we have constructed f_n with the following properties: $dom(f_n) = \{a_{i_1}, ..., a_{i_m}\}$, the image of $f_n = \{b_{j_1}, ..., b_{j_m}\}$, $f_n(a_{i_l}) = b_{j_l}$, $a_{i_1} < a_{i_2} < ... < a_{i_m}$, and f_n is *order preserving*. We now construction f_{n+1} in two steps:
    (a) Step 1: Suppose that k is the smallest index such that $a_k \notin dom(f_n)$. Then one of the following is true
        (i) $a_k < a_{i_l}$ for every $l \in \{1, ..., m\}$.
        (ii) There exists some $l \in \{1, ..., m-1\}$ such that $a_{i_l} < a_k < a_{i_{l+1}}$.
        (iii) $a_k > a_{i_l}$ for every $j \in \{1, ..., m\}$.
    We work with case 2: Suppose that $a_{i_l} < a_k < a_{i_{l+1}}$ for some l. Notice that by the density axiom, $N \models \exists x(b_{j_l} < x < b_{j_{l+1}})$. Hence there exists some $t \in \mathbb{N}$ such that $N \models b_{j_l} < b_t < b_{j_{l+1}}$. Let $g_{n+1} \supset f_n$ and also $g_{n+1}(a_k) = b_t$. Hence $dom(g_{n+1}) = dom(f_n) \cup \{a_k\}$.
    (b) Step 2: Suppose that r is the smallest index such that $b_r \notin$ the image of g_{n+1}. By a similar processes as before, we find some $a_s \in \mathbb{Q}$ such that the order type of a_s over the domain of g_n is the same as the order type of b_r over the image of g_{n+1}. We let $f_{n+1} \supset g_{n+1}$ where $f_{n+1}(a_s) = b_r$. (We need this second argument to ensure that the function we build is surjective).
(5) We let $f = \bigcup_{n \geq 1} f_n$. We claim that f is an isomorphism. $\square$

Definition 2.9. Let $\mathcal{L}$ be a first order language and let $\mathcal{M}$ be the collection of all $\mathcal{L}$-structures. We let $I(T, \kappa) = |\{[M] \in \mathcal{M}/\cong : M \models T\}|$

Example 2.10. Notice that if T is countable categorical, then $I(T, \aleph_0) = 1$.

Example 2.11. Consider the language $\mathcal{L} = \{P_1, P_2\}$ where P_1, P_2 are both unary predicates. Suppose that T is the following theory:

(1) $\forall x(P_1(x) \vee P_2(x))$.
(2) $\forall x(P_1(x) \leftrightarrow \neg P_2(x))$.
(3) For each $n \geq 1$ and $k \in \{1, 2\}$, $\exists x_1, ..., x_n \left(\bigwedge_{1 \leq i \neq j \leq n} x_i \neq x_j \wedge \bigwedge_{i=1}^{n} P(x_i) \right)$.

Then we claim that $I(T, \aleph_0) = 1, I(T, \aleph_1) = 3, I(T, \aleph_2) = 5, ..., I(T, \aleph_\omega) = \aleph_0$.

3. Structures

Definition 3.1. Let $M_1 = (A_1; ...)$ and $M_2 = (A_2; ...)$ be $\mathcal{L}$-structures. Then we say that M_1 is a substructure of M_2 if

(1) $A_1 \subseteq A_2$.
(2) The restriction of the interpretation any relation, function, or constant symbol from M_2 to M_1 is precisely the interpretation of that relation, function, or constant in M_1. In other words;
   (a) For any n-ary function symbol f, we have that $f^{M_2}|_{A_1^n} = f^{M_1}$.
   (b) For any n-ary relational symbol R, we have that $R^{M_2} \cap A_1^n = R^{M_1}$.
   (c) For any constant symbol c, we have that $c^{M_2} = c^{M_1}$.