PKU MODEL THEORY NOTES

KYLE GANNON

Proposition 0.1. *Suppose that T is a complete theory and $M, N \models T$. Then $M \equiv N$.*

Proof. Suppose not. Then there exists a sentence such that $M \models \varphi$ and $N \models \neg\varphi$. Since T is complete, $T \vdash \varphi$ or $T \vdash \neg\varphi$. Suppose that $T \vdash \varphi$. Then $N \models T$ and so $N \models \varphi$, but this is a contradiction. □

Remark 0.2. We remark that $(\mathbb{Q}, \leq) \equiv (\mathbb{R}, \leq)$. We showed in the previous week's notes that $T_{\leq} :=$ Dense linear orderings without endpoints was countably categorical. It is also true that this theory has no finite models. Hence $T_{\leq}$ is complete by Vaught's test. We have that $(\mathbb{R}, \leq) \models T_{\leq}$ and $(\mathbb{Q}, \leq) \models T_{\leq}$. Hence by the previous proposition, $(\mathbb{Q}, \leq) \equiv (\mathbb{R}, \leq)$.

1. Elementary extension and types

Definition 1.1. Let $M = (A; ...)$ and $N = (B; ...)$ be $\mathcal{L}$-structures. We say that M is a substructure of N if

(1) $A \subseteq B$.
(2) For every n-ary relation R in $\mathcal{L}$, $R^M = R^N \cap A^n$.
(3) For every n-ary function symbols f in $\mathcal{L}$, $f^M = f^N|_{A^n}$.
(4) For every constant symbol c, $c^M = c^N$.

Definition 1.2. Suppose that $M = (A; ...)$ and $N = (B; ...)$ are $\mathcal{L}$-structures. We say that $M \preceq N$ if

(1) $A \subseteq B$.
(2) For any formula $\psi(x_1, ..., x_n)$ and $a_1, ..., a_n \in M$,

$$M \models \psi(a_1, ..., a_n) \iff N \models \psi(a_1, ..., a_n).$$

Proposition 1.3. *Let M and N be $\mathcal{L}$-structures such that $M \preceq N$. Then $M \equiv N$.*

Proof. Induction. □

Proposition 1.4. *Suppose that M is an $\mathcal{L}$-structure. Then there exists some $\mathcal{L}$-structure N such that $M \neq N$ and $M \prec N$.*

Proof. Compactness. Let $\mathcal{L}_M = \mathcal{L} \cup \{c_m : m \in M\}$. Turn M into a $\mathcal{L}_M$ structure. If $N \models Th_{\mathcal{L}_M}(M)$, then $M \prec N$ (provided the domain of M is a subset of the domain of N - but this is a non-issue). □

Definition 1.5. Let $M = (A; ...)$ be an $\mathcal{L}$-structure. Let $B \subseteq A$ and $\overline{x} = x_1, ..., x_n$. We let $\mathcal{L}_{\overline{x}}(B) = \{\varphi(x_1, ..., x_n, b_1, ..., b_m) : \varphi(x_1, ..., x_n, y_1, ..., y_m)$ is an $\mathcal{L}$-formula, $b_1, ..., b_m \in B\}$.

Definition 1.6. Fix a model $M = (A; ...)$, $B \subseteq A$, and $\overline{x} = x_1, ..., x_n$ a tuple of variables. A *partial* type π (in $\overline{x}$ over B) is defined as follows:

(1) $\pi \subset \mathcal{L}_{\overline{x}}(B)$.
(2) π is finitely satisfiable, i.e. for any $\pi_0 \subseteq_{finite} \pi$, there exists some $(a_1, ..., a_n) \in A^n$ such that for any $\theta(x_1, ..., x_n) \in \pi_0$, $M \models \theta(a_1, ..., a_n)$.

Moreover, we say that π is complete if for any $\theta(x_1, ..., x_n) \in \mathcal{L}_{\overline{x}}(B)$, either $\theta(x_1, ..., x_n) \in \pi$ or $\neg\theta(x_1, ..., x_n) \in \pi$. Finally, we let $S_{\overline{x}}(B)$ be the collection of complete types (in $\overline{x}$ over B). If $B = A$, we write $S_{\overline{x}}(B)$ simply as $S_{\overline{x}}(M)$.

Example 1.7. Consider $(\mathbb{N}; \leq)$. Let $\pi = \{n \leq x : n \in \mathbb{N}\}$. Then π is a type in x over $\mathbb{N}$.

Definition 1.8. Fix a model M and let $B \subseteq M$. Let π be a type in variable x over B (i.e. $\pi \subseteq \mathcal{L}_x(B)$ and π is finitely satisfiable). Let $a \in M$. We say that $a \models \pi$ or a realizes π for for every $\theta(x) \in \pi$, $M \models \theta(a)$.

Proposition 1.9. *Let $M \prec N$ and consider $p = \{\varphi(x) \in \mathcal{L}_x(M) : N \models \varphi(b)\}$. Then $p \in S_x(M)$, i.e. p is a complete type.*

Proof. First, we want to show that p is a type.

(1) It is clear that $p \subseteq \mathcal{L}_x(M)$.
(2) We need to show that p is finitely satisfiable. Let $p_0 \subseteq_{finite} p$. Then $p_0 = \{\varphi_1(x, \bar{c}_1), ..., \varphi_n(x, \bar{c}_n)\}$. Let $\theta(x, \bar{c}) = \bigwedge_{i=1}^{n} \varphi_i(x, \bar{c}_i)$. Then $N \models \theta(b, \bar{c})$ and so $N \models \exists x\theta(x, \bar{c})$. Since $M \prec N$, we have that $M \models \exists x\theta(x, \bar{c})$. Hence there exists some $d \in M$ such that $M \models \theta(d, \bar{c})$. By unpacking, we see that $M \models \varphi_i(d, \bar{c}_i)$ for any $\varphi_i(x, \bar{c}_i) \in p_0$. Hence p_0 is satisfied in M. Hence p is finitely satisfiable.
(3) Finally, we argue that p is complete. Let $\theta(x) \in \mathcal{L}_x(M)$. Then $\theta(x) \in \mathcal{L}_x(N)$. So $N \models \theta(b)$ or $N \models \neg\theta(b)$. By construction, $\theta(x) \in p$ or $\neg\theta(x) \in p$ and so we are finished. $\square$

Proposition 1.10. *Let M be an $\mathcal{L}$-structure. Let $p \in S_x(M)$. Then there exists some N and $b \in N$ such that $b \models p$.*

Proof. We provide a sketch. Consider $\mathcal{L}_1 = \mathcal{L} \cup \{c_a : a \in M\}$. We let M_1 be the $\mathcal{L}_1$-structure where the interpretation of each symbol in $\mathcal{L}$ is the same as in M and $c_a^{M_1} = a$. We let $D(M_1) = Th_{\mathcal{L}_1}(M_1)$. Notice that p can be written as a type in x over $\emptyset$ in $\mathcal{L}_1$, call it p_1. In particular, $p_1 = \{\varphi(x, c_{a_1}, ..., c_{a_n}) : \varphi(x, a_1, ..., a_n) \in p\}$. Now consider $\mathcal{L}_2 = \mathcal{L}_1 \cup \{d\}$ where d is a new constant symbol. Consider $T = D(M_1) \cup \{\theta(d) : \theta(x) \in p_1\}$. We claim that this is finitely consistent. Let $M_2 \models T$. Let $(M_2)_*$ be the $\mathcal{L}$-structure be obtained by forgetting constants and let d_* be the element in $(M_2)_*$ such that $M_2 \models d = d_*$. We[1] have that $M \prec M_2$ and the element $d_* \models p$. $\square$

Theorem 1.11 (Tarski-Vaught)**.** *Let M be a substructure of N. Then the following are equivalent.*

(1) $M \preceq N$.
(2) For any $a_1, ..., a_n$ in M and formula $\varphi(x, y_1, ..., y_n)$, if there exists a b in N such that $N \models \varphi(b, a_1, ..., a_n)$, then there exists some $d \in M$ such that $N \models \varphi(d, a_1, ..., a_n)$.

[1] To be very precise, also has to possibly change the underlying set of M_2 so that $M \subset (M_2)_*$.

Proof. ($\Rightarrow$) Suppose that $M \preceq N$. Suppose that $\exists b \in N$ such that $N \models \varphi(b, a_1, ..., a_n)$. Then $N \models \exists x \varphi(x, a_1, ..., a_n)$. By elementary, $M \models \exists x \varphi(x, a_1, ..., a_n)$. Hence $M \models \varphi(d, a_1, ..., a_n)$ for some $d \in A$. By elementary, $N \models \varphi(d, a_1, ..., a_n)$.

($\Leftarrow$) Suppose the condition holds. This follows from the induction on complexity of formulas. This should be checked in the privacy of your own home. $\square$

Proposition 1.12. *Suppose that $M = (\mathbb{N}, \leq)$. If $N \preceq M$, then $N = M$.*

Proposition 1.13. $(2\mathbb{Z}; 0, +)$ *is **not** an elementary substructure of* $(\mathbb{Z}; 0, +)$

2. Downward Löwenheim-Skolem theorem

When do elementary substructures exist?

Proposition 2.1. *Suppose that $|\mathcal{L}| \leq \aleph_0$ and $|M| = \kappa$ where $\kappa > \aleph_0$. Then there exists a model N such that*

(1) $N \prec M$.
(2) $|N| = \aleph_0$

Proof. For every $\mathcal{L}$-formula $\varphi(x, y_1, ..., y_n)$, we define the partial function $f_\varphi : M^n \to M$ where if $M \models \exists x \varphi(x, a_1, ..., a_n)$, then $M \models \varphi(f_\varphi(a_1, ..., a_n), a_1, ..., a_n))$. So, the domain of f_φ is $\{(a_1, ..., a_n) \in M^n : M \models \exists x \varphi(x, a_1, ..., a_n)\}$.

Let $B \subset M$ such that $B \neq \emptyset$. We let $F(B) = \{f_\varphi(b_1, ..., b_n) : b_1, ..., b_n \in B, \varphi(x, y_1, ..., y_n) \text{ is an } \mathcal{L}\text{-formula}\}$. We let $F^{n+1} = F(F^{n+1}(B))$ and $F^\omega = \bigcup_{i \in \mathbb{N}} F^n(B)$. We now turn F^ω into an $\mathcal{L}$-structure and apply Tarski-Vaught. For ease of notation, we let $C = F^\omega(B)$ and construct the structure $N = (C; ...)$.

(1) Let c be a constant symbol. We let $c^N = c^M$. How do we know that $c^M \in N$? Consider the formula $\theta(x, y) := x = c \wedge y = y$. Then $f_\theta(b) = c^M$ for any $b \in B$. Hence $c^M \in F(B) \subset N$.
(2) Let R be an n-ary relation symbol. We let $R^N = R^M|_{C^n}$
(3) Let f be an n-ary function symbol. Again, we let $f^N = f^M|_C$. We also need to check that this is well-defined. Suppose that $a_1, ..., a_n \in N$. Then there exists some k such that $a_1, ..., a_n \in F^k(B)$. Consider the formula $\theta(x, y_1, ..., y_n) := f(y_1, ..., y_n) = x$. Hence $f_\theta(a_1, ..., a_n) \in F^{k+1}(B)$ and so $f_\theta(a_1, ..., a_n) \in N$. Thus our function is well-defined.

We claim that $N \prec M$. We apply Tarski-Vaught. Let $\varphi(x, y_1, ..., y_n)$ be any $\mathcal{L}$-formula, $a_1, ..., a_n \in N$ and $d \in M$ such that $M \models \varphi(d, a_1, ..., a_n)$. Then $M \models \exists x \varphi(x, a_1, ..., a_n)$. Hence $(a_1, ..., a_n)$ is in the domain of f_φ. Moreover, there exists some k such that $a_1, ..., a_n \in F^k(B)$. Hence $f_\varphi(a_1, ..., a_n) \in F^{k+1}(B)$ and so $f_\varphi(a_1, ..., a_n) \in N$. By definition, we know that $M \models \varphi(f_\varphi(a_1, ..., a_n), a_1, ..., a_n)$ and so the proof is complete.

Now suppose that $|B| \leq \aleph_0$. Then $|F(B)| \leq \aleph_0$ and moreover $|F^\omega(B)| \leq \aleph_0$. Therefore, choosing B to be countable gives us a model N such that $|N| = \aleph_0$. $\square$

The following theorem is the generalized version of the Löwenheim-Skolem theorem. The downward version is similar to what we proved above (almost identical). The upward version follows via compactness/completeness.

Theorem 2.2. *Let $\mathcal{L}$ be a language and M an $\mathcal{L}$-structure. Suppose that $|\mathcal{L}| = \kappa$ and M is an infinite $\mathcal{L}$-structure where $|M| = \lambda$ (κ and λ are infinite cardinals).*

(1) (Upward version) For any $\mu \geq \max\{\kappa, \lambda\}$, there exists a model N such that $M \preceq N$ and $|N| = \mu$.

(2) *(Downward version) For any* μ *such that* $\kappa \leq \mu \leq \lambda$*, there exists a model* N *such that* $N \preceq M$ *and* $|N| = \mu$*.*

In particular, if $|\mathcal{L}| \leq \aleph_0$ *and* $|M| > \aleph_0$*,*

(1) *For any* $\kappa \geq |M|$*, there exists* N *such that* $M \preceq N$ *and* $|N| = \kappa$*.*
(2) *There exists some* N *such that* $|N| = \aleph_0$ *and* $N \preceq M$*.*