PKU MODEL THEORY NOTES

KYLE GANNON

1. Ultrafilters

Definition 1.1. Let I be an indexing set and $\mathcal{P}(I)$ denote the power set of I. A filter $\mathcal{F}$ (on I) is a non-empty subset of $\mathcal{P}(I)$ with the following properties:

(1) $\emptyset \not\in \mathcal{F}$.
(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
(3) If $B \supseteq A$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

The following facts are easy to check.

Fact 1.2. *Let I be an indexing set and $\mathcal{F}$ is a filter on I.*

(1) For any finite collections $A_1, ..., A_n \in \mathcal{F}$, $\bigcap_{i=1}^{n} A_i \in \mathcal{F}$.
(2) $I \in \mathcal{F}$.

Example 1.3. Let $I = \mathbb{N}$.

(1) For any $a \in \mathbb{N}$, we let $D_a = \{X \subseteq \mathbb{N} : a \in X\}$. D_a is a filter. Filters of this form are called *principal*.
(2) Let $\mathcal{F}_{cofinite} = \{X \subseteq \mathbb{N} : |\mathbb{N}\backslash X| < \aleph_0\}$. This is a filter and is known as the cofinite-filter or the *Frechet* filter.

Definition 1.4. Let I be an indexing set and let $\mathcal{F}$ be a filter on I. We say that F is an ultrafilter on I if for every $X \subseteq I$, either $X \in \mathcal{F}$ or $I\backslash X \in \mathcal{F}$.

Proposition 1.5. *Suppose that $\mathcal{F}$ is a filter on I. Suppose that $A \subset \mathbb{N}$ such that $A \not\in \mathcal{F}$ and $I\backslash A \not\in \mathcal{F}$. Let $\mathcal{F}_A = \{B \cap A : B \in \mathcal{F}\}$. Let $\overline{\mathcal{F}_A} = \{C : \exists E \in \mathcal{F}_A$ such that $C \supseteq E\}$. Then*

(1) $F \subset \overline{\mathcal{F}_A}$.
(2) $\overline{\mathcal{F}_A}$ is a filter on I.

Proof. Suppose that $C \in \mathcal{F}$. Then $C \supseteq C \cap A$ and so $C \in \overline{\mathcal{F}_A}$. Moreover, $A \in \overline{\mathcal{F}_A}$, but $A \not\in F$ by assumption. Hence $F \subsetneq \overline{\mathcal{F}_A}$.

We now show that $\overline{\mathcal{F}_A}$ is a filter.

(1) Suppose that $\emptyset \in \overline{\mathcal{F}_A}$. Then $\emptyset \in \mathcal{F}_A$. Hence there exists some $B \in \mathcal{F}$ such that $B \cap A = \emptyset$. Then $I\backslash A \supseteq B$ which implies that $I\backslash A \in F$. This contradicts our assumption.
(2) Suppose that $C_1, C_2 \in \overline{\mathcal{F}_A}$. Then there exists $B_1, B_2 \in \mathcal{F}$ such that $C_1 \supseteq B_1 \cap A$ and $C_2 \supseteq B_2 \cap A$. Then $B_1 \cap B_2 \in \mathcal{F}$ and $C_1 \cap C_2 \supseteq (B_1 \cap B_2) \cap A$. By construction, $C_1 \cap C_2 \in \overline{\mathcal{F}_A}$.
(3) Suppose that $C_1 \in \overline{\mathcal{F}_A}$ and $C_2 \supseteq C_1$. Then there exists $B \in \mathcal{F}$ such that $C_1 \supseteq B \cap A$. So $C_2 \supseteq B \cap A$ and therefore $C_2 \in \overline{\mathcal{F}_A}$.

Hence $\overline{\mathcal{F}_A}$ is a filter. □

Theorem 1.6. *Let I be an indexing set and suppose that $\mathcal{F}$ is a filter on I. Then there exists an ultrafilter D on I such that $\mathcal{F} \subseteq D$.*

Proof. This follows from an application of Zorn's lemma. Consider $(\mathcal{G}, \subseteq)$ where $\mathcal{G} = \{D : D \text{ is a filter over } I \text{ and } D \supseteq \mathcal{F}\}$. Let $(\mathcal{C}, \subseteq)$ be a chain in this partial order. We need to show that this chain has an upper bound. Consider the set $H = \bigcup_{D \in \mathcal{C}} D$. We claim that $H \in \mathcal{G}$ and H is an upper bound for C. It suffices to show that H is a filter.

(1) Suppose that $\emptyset \in H$. Then there exists some $D \in \mathcal{C}$ such that $\emptyset \in D$, but this is a contradiction since D is a filter. Hence $\emptyset \notin H$.
(2) Suppose that $A_1, A_2 \in H$. Then there exists some $D \in \mathcal{C}$ such that $A_1, A_2 \in D$. Then $A_1 \cap A_2 \in D$ which implies $A_1 \cap A_2 \in H$.
(3) Suppose that $A_1 \supseteq A_2$ and $A_2 \in H$. Then there is some $D \in \mathcal{C}$ such that $A_2 \in D$. So $A_1 \in D$ and so $A_1 \in H$.

By Zorn's lemma, there exists a maximal element $K \in \mathcal{G}$. Since $K \in \mathcal{G}$, we know that $F \subseteq K$. We claim that K is an ultrafilter. Assume not. Then there exists some $A \subseteq I$ such that $A \notin K$ and $I \backslash A \notin K$. By the previous proposition $\overline{K_A}$ is a filter which properly extends K. Thus K is not maximal and so we have a contradiction. □

Definition 1.7. Let I be an indexing set and D be an ultrafilter on I. We say that D is a principle ultrafilter if there exists some $i \in I$ such that $D = D_i = \{X \subseteq I : i \in X\}$. Otherwise, we say that D is non-principle.

Proposition 1.8. *Suppose that I is finite. Then every ultrafilter on I is principle.*

Proposition 1.9. *Suppose that I is infinite. Then there exists a non-principle ultrafilter on I. Moreover, if D is a non-principle ultrafilter on I, then D contains every cofinite set, i.e. for any $X \subseteq I$ such that $|I \backslash X| < \aleph_0$, $X \in D$.*

2. Ultraproducts

Definition 2.1. Let I be an indexing set and $(M_i)_{i \in I}$ an indexed family of $\mathcal{L}$-structures. We consider the product $\prod_{i \in I} M_i$. Notice that every element in $\prod_{i \in I} M_i$ can be thought of as a function $f : I \to \bigcup_{i \in I} M_i$ where $f(i) \in M_i$. Elements of $\prod_{i \in I} M_i$ can also be thought of as sequences of points $(a_1, a_2, a_3, ...)$ where each $a_i \in M_i$.

Now let D be a filter on I. We define a relation $\sim_D$ on $\prod_{i \in I} M_i$ where $f \sim_D g$ if and only if $\{i \in I : f(i) = g(i)\} \in D$.

Proposition 2.2. *Let I be an indexing set, $(M_i)_{i \in I}$ an indexed family of $\mathcal{L}$-structures, and D be a filter on I. Then $\sim_D$ is an equivalence relation on $\prod_{i \in I} M_i$.*

Proof. Exercise. □

Definition 2.3. Let I be an indexing set, $(M_i)_{i \in I}$ an indexed family of $\mathcal{L}$-structures, and D be an ultrafilter on I. We let $\prod_D M_i = \prod_{i \in I} M_i / \sim_D$. In other words, if $[f]_D = \{g \in \prod_{i \in I} M_i : f \sim_D g\}$, then $\prod_D M_i = \{[f]_D : f \in \prod_{i \in I} M_i\}$. $\prod_D M_i$ is an $\mathcal{L}$-structure with the following interpretations of $\mathcal{L}$-symbols (for ease of notation, we let $N = \prod_D M_i$):

(1) Let R be an n-ary relation symbol. Then $([f_1]_D, ..., [f_n]_D) \in R^N$ if and only if $\{i \in I : (f_1(i), ..., f_n(i)) \in R^{M_i}\} \in D$.

(2) Let G be an n-ary function symbols. Then $G^N([f_1]_D, ..., [f_n]_D) = [h]_D$ where $h(i) = G^{M_i}(f_1(i), ..., f_n(i))$.
(3) Let c be a constant symbol. Then $c^N = [f_c]_D$ where $f_c(i) = c^{M_i}$.

The structure $\prod_D M_i$ is called an ultraproduct.

Theorem 2.4 (Loś's Theorem)**.** . *Let I be an indexing set, D an ultrafilter on I, $(M_i)_{i\in I}$ an indexed family of $\mathcal{L}$-structures, and $f_1, ..., f_n \in \prod_{i\in I} M_i$. Then for any $\mathcal{L}$-formula $\varphi(x_1, ..., x_n)$,*

$$\prod_D M_i \models \varphi([f_1]_D, ..., [f_n]_D) \iff \{i \in I : M_i \models \varphi(f_1(i), ..., f_n(i))\} \in D.$$

Moreover, for any $\mathcal{L}$-sentence φ, we have that

$$\prod_D M_i \models \varphi \iff \{i \in I : M_i \models \varphi\} \in D.$$

Proof. Induction hypothesis: Suppose the condition holds for $\psi(x_1, ..., x_n)$ and $\theta(x_1, ..., x_n)$.

Conjunction: Follows from intersection part.

$$\begin{aligned}
&\prod_D M_i \models \psi([f_1]_D, ..., [f_n]_D) \wedge \theta([f_1]_D, ..., [f_n]_D) \\
&\iff \prod_D M_i \models \psi([f_1]_D, ..., [f_n]_D) \text{ and } \prod_D M_i \models \theta([f_1]_D, ..., [f_n]_D) \\
&\iff \{i \in I : M_i \models \psi(f_1(i), ..., f_n(i))\} \in D \text{ and } \{i \in I : M_i \models \psi(f_1(i), ..., f_n(i))\} \in D \\
&\iff \{i \in I : M_i \models \psi(f_1(i), ..., f_n(i))\} \cap \{i \in I : M_i \models \psi(f_1(i), ..., f_n(i))\} \in D \\
&\iff \{i \in I : M_i \models \psi(f_1(i), ..., f_n(i)) \wedge \psi(f_1(i), ..., f_n(i))\} \in D
\end{aligned}$$

Negation: Check.

Existential quantifier: We want to show that the statement holds for $\exists x_1 \psi(x_1, ..., x_n)$.

$$\begin{aligned}
\prod_D M_i \models \exists x_1 \psi(x_1, [f_2]_D, ..., [f_n]_D) &\iff \prod_D M_i \models \psi([g]_D, [f_2]_D, ..., [f_n]_D) \\
&\iff \{i \in I : M_i \models \psi(g(i), f_2(i), ..., f_n(i))\} \in D \\
&\iff \{i \in I : M_i \models \exists x \psi(x, f_2(i), ..., f_n(i))\} \in D.
\end{aligned}$$

Last if and only if forward direction is trivial, backwards direction "relies on the Axiom of choice". □

Definition 2.5. Let I be an indexing set, $M_i = M$ for every $i \in I$, and D be an ultrafilter on I. Then $\prod_D M_i$ is called an ultrapower and there is a natural map $\Delta : M \to \prod_D M$ via $\Delta(a) = [(a_0, a_1, a_2, ...)]_D$. In other words, an element a in M is mapped to equivalence class of the constant function f_a where for any $i \in I$, $f_a(i) = a$.

Example 2.6. Let $I = \omega$, $M_i = (\mathbb{N}, <)$ for each $i < \omega$, and D be an ultrafilter on I which extends the confinite filter. Then

(1) We notice that there are elements in $\prod_D M_i$ which are larger than every standard natural numbers, e.g. $[(0, 1, 2, 3, 4, ...)]_D$.

(2) $\prod_D M_i$ has no greatest elements since $\{i \in I : M_i \models \forall x \exists y(x < y)\} \in D$.
(3) We notice that $\prod_D M_i$ is not well-ordered: Consider the sequence

$$[(0,1,2,3,4,5,...)]_D > [(0,0,1,2,3,4,...)]_D > [(0,0,0,1,2,3)]_D...$$

Example 2.7. Consider the theory of algebraically closed fields of characteristic p (ACF_p) in the language $\mathcal{L}_{ring} = \{+, \times, 0, 1\}$. These theories say

(1) The structure is a field.
(2) The structure is algebraically closed (i.e., every polynomial has a solution).
(3) $\underbrace{1 + ... + 1}_{p-times} = 0$

For each prime p, we let $\mathbf{F}_p \models ACF_p$. Then $\prod_D \mathbf{F}_p \models ACF_0$. More generically, one can prove that $\prod_D \mathbf{F}_p \equiv (\mathbb{C}; +, \times, 0, 1)$.