PKU MODEL THEORY NOTES

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1. Quantifier Elimination

Definition 1.1. Let T be an $\mathcal{L}$-theory. T admits quantifier elimination if for any formula $\varphi(x_1, ..., x_n)$, there exists a quantifier free formula $\psi_\varphi(x_1, ..., x_n)$ such that $T \vdash \forall x_1 ... \forall x_n(\varphi(x_1, ..., x_n) \leftrightarrow \psi_\varphi(x_1, ..., x_n))$.

Definition 1.2. A literal is an atomic formula or the negation of an atomic formula.

Fact 1.3. *Let T be an $\mathcal{L}$-theory and $\psi(x_1, ..., x_n)$ be a quantifier free formula. Then $\psi(x_1, ..., x_n)$ is equivalent to $\bigvee_{i=1}^{n} \bigwedge_{j_i=1}^{m_i} \alpha_{j_i}$ where each α_i is a literal.*

Lemma 1.4. *Let T be an $\mathcal{L}$-theory. Suppose that for every quantifier free formula $\psi(x, y_1, ..., y_n)$ of the form $(\alpha_1 \wedge ... \wedge \alpha_m)$ where each α_i is a literal, there exists a quantifier free formula $\psi_\varphi(y_1, ..., y_n)$ such that $T \vdash \forall y_1, ..., y_n(\exists x \varphi(x, y_1, ..., y_n) \leftrightarrow \psi_\varphi(y_1, ..., y_n))$. Then T admits quantifier elimination.*

Proof. We want to show that all formulas are equivalent to quantifier free formulas, given the hypothesis above. Base Case: Every atomic formula is equivalent to a quantifier free formula (namely itself).

Induction Hypothesis: Suppose that $\theta(x_1, ..., x_n)$ and $\chi(x_1, ..., x_n)$ are formulas such that

$$T \vdash \forall x_1 ... \forall x_n(\theta(x_1, ..., x_n) \leftrightarrow \psi_\theta(x_1, ..., x_n)),$$

and,

$$T \vdash \forall x_1 ... \forall x_n(\chi(x_1, .., x_n) \leftrightarrow \psi_\chi(x_1, ..., x_n))$$

where ψ_θ and ψ_χ are quantifier free.

Induction step:

1. **Negation:** $\neg\theta(x_1, ..., x_n)$ is equivalent to $\neg\psi_\theta(x_1, ..., x_n)$.
2. **Conjunction:** $\theta(x_1, ..., x_n) \wedge \chi(x_1, ..., x_n)$ is equivalent to $\psi_\theta(x_1, ..., x_n) \wedge \psi_\chi(x_1, ..., x_n)$.
3. **Existential quantification:** Consider $\exists x_1 \theta(x_1, ..., x_n)$. Notice that

$$\exists x_1 \theta(x_1, ..., x_n) \overset{(a)}{\equiv} \exists x_1 \psi_\theta(x_1, ..., x_n) \overset{(b)}{\equiv} \exists x_1 \left(\bigvee_{i=1}^{n} \bigwedge_{j_i=1}^{m_i} \alpha_{j_i} \right)$$

$$\overset{(c)}{\equiv} \left(\bigvee_{i=1}^{n} \exists x_1 \bigwedge_{j_i=1}^{m_i} \alpha_{j_i} \right) \overset{(d)}{\equiv} \bigvee_{i=1}^{n} \gamma_i$$

We provide the following details/justifications:

(a) Induction hypothesis.

(b) Fact 1.3, $\psi_\theta(x_1,...,x_n)$ is equivalent to $\bigvee_{i=1}^{n}\bigwedge_{j_i=1}^{m_i}\alpha_{j_i}$ where each α_i is a literal.
(c) Existential quantification commutes with disjunction.
(d) By our assumption, $\exists x_1 \bigwedge_{j=1}^{m_i}\alpha_i$ is equivalent to γ_i where γ_i is a quantifier free formula. □

Proposition 1.5. *Suppose M is a substructure of N. For any quantifier-free formula, $\varphi(x_1,...,x_n)$ and $a_1,...,a_n \in M$, $M \models \varphi(a_1,...,a_n) \iff N \models \varphi(a_1,...,a_n)$.*

Proof. For any term $t = t(x_1,...,x_n)$ where the variable of t are among $\{x_1,...,x_n\}$, we let $t^M(a_1,...,a_n)$ be the unique element in M given by plugging a_i in for x_i for each variable in t. We claim that $t^M(a_1,...,a_n) = t^N(a_1,...,a_n)$ for any $a_1,...,a_n \in M$.

(1) If t is x, then $t^M(a) = a = t^N(a)$.
(2) If t is c, then $c^M = c^N$ (by substructure).
(3) Suppose that $t_1,...,t_m$ are terms and $t_i^M(a_1,...,a_n) = t_i^N(a_1,...,a_n)$ for each $i \le m$. Let f be an m-ary function and consider $t = f(t_1,...,t_m)(x_1,...,x_n)$. Then

$$\begin{aligned} t^M(a_1,...,a_n) &= f^M(t_1^M(a_1,...,a_n),...,t_m^M(a_1,...,a_n)) \\ &= f^N(t_1^M(a_1,...,a_n),...,t_m^M(a_1,...,a_n)) \\ &= f^N(t_1^N(a_1,...,a_n),...,t_m^N(a_1,...,a_n)) = t^N(a_1,...,a_n). \end{aligned}$$

where the second equality follows from the definition of substructure.

Base Case: Let $\varphi(x_1,...,x_n)$ be an atomic formula. So $\varphi(x_1,...,x_n)$ is $R(t_1,...,t_m)(x_1,...,x_n)$. Now

$$\begin{aligned} M \models \varphi(a_1,...,a_n) &\iff M \models R(t_1,...,t_m)(a_1,...,a_n) \\ &\iff (t_1^M(a_1,...,a_n),...,t_m^M(a_1,...,a_n)) \in R^M \\ &\iff (t_1^M(a_1,...,a_n),...,t_m^M(a_1,...,a_n)) \in R^N \\ &\iff (t_1^N(a_1,...,a_n),...,t_m^N(a_1,...,a_n)) \in R^N \\ &\iff N \models R(t_1,...,t_m)(a_1,...,a_n) \iff N \models \varphi(a_1,...,a_n). \end{aligned}$$

Induction Hypothesis: Suppose that $\theta(x_1,...,x_n)$ and $\chi(x_1,...,x_n)$ are quantifier free formulas such that for any $a_1,...,a_n \in M$, $M \models \theta(a_1,...,a_n) \iff N \models \theta(a_1,...,a_n)$ and $M \models \chi(a_1,...,a_n) \iff N \models \chi(a_1,...,a_n)$.

Induction Step: Need to check only negation and conjunction. Both cases are straightforward. □

Theorem 1.6. *Suppose that $M, N \models T$, M is a substructure of N and T admits quantifier elimination. Then $M \preceq N$.*

Proof. Fix a formula $\varphi(x_1,...,x_n)$ and $a_1,...,a_n \in M$. By quantifier elimination, we have that $T \vdash \forall x_1,...,x_n(\varphi(\bar{x}) \leftrightarrow \psi_\varphi(\bar{x}))$ where $\psi_\varphi(\bar{x})$ is quantifier free. Then we have that

$$\begin{aligned} N \models \varphi(a_1,...,a_n) &\implies N \models \psi_\varphi(a_1,...,a_n) \\ &\implies M \models \psi_\varphi(a_1,...,a_n) \\ &\implies M \models \varphi(a_1,...,a_n), \end{aligned}$$

where the first and third implication follows from the fact that $N, M \models T$. The second implication follows from the fact that M is a substructure of N. □

Example 1.7. Let T be the theory of dense linear orderings without endpoints in the language $\mathcal{L} = \{<\}$. Then T admits quantifier elimination.

Proof. Consider the formula φ which is $\exists x(\beta_1 \wedge ... \wedge \beta_n)$ where each β_i is a literal. Notice that each atomic formula is of the form $u = v$ or $u < v$.

(1) Negation elimination:
   (a) If $\beta_j = \neg(u < v)$, replace with $(u = v) \vee (v < u)$.
   (b) If $\beta_j = \neg(u = v)$, replace with $(u < v) \vee (v < u)$.

(2) (check): After replacing the β_i's above, we can rewrite $\exists x(\beta_1 \wedge ... \wedge \beta_n)$ as $\exists x \bigvee_{i=1}^{n} \bigwedge_{j_i=1}^{n} \alpha_{i_j}$ where each α_{i_j} is an atomic formula. This formula is equivalent to $\bigvee_{i=1}^{n} \exists x(\bigwedge_{j_i=1}^{m_i} \alpha_{i_j})$. Hence it suffices to show that $\exists x(\bigwedge_{j_i} \alpha_{i_j})$ is equivalent to a quantifier free formula.

(3) Let $\gamma_i = (\bigwedge_{j_i=1}^{m_i} \alpha_{i_j})$. By the previous bullet, it suffices to show that $\exists x \gamma_i$ is equivalent to a quantifier free formula. We re-index γ and write it simply as $\bigwedge_{i=1}^{m} \alpha_i$. We now give an algorithm to convert γ to a quantifier free formula.
   (a) Check each α_i: If α_i is of the form $x = x$, we can remove it. Then move to the next step.
   (b) Check each α_i. If α_i does does not contain x, we can move it outside the quantifier. If no α_i contains x, we can simply remove the quantifier and return "finished". If we are not finished, move to the next step.
   (c) Check each α_i: If there exists α_i of the form $x = y$, we can replace every instance of x in γ_i with y and return "finished". If we are not finished, move to the next step.
   (d) After doing the above, the remaining literals under the scope of our quantifier are of the form $x < x$, $y < x$ and $x < y$.
      (i) If there exists some α_i which is of the form $x < x$, we replace γ with $\bigwedge_{y \in F(\gamma)} y \neq y$ where $F(\gamma)$ are the free variables which occur in γ.
      (ii) We are left with the case where $\gamma = \bigwedge \eta \wedge \exists x(\bigwedge_{j \leq t} y_j < x \wedge \bigwedge_{l \leq k} x < y_l)$ where η is a collection of atomic formulas without any instance of x (this is the portion we move outside the quantifier from step (b)). We claim that

$$\exists x \gamma \equiv \bigwedge \eta \wedge (\bigwedge_{j \leq t, l \leq k} y_j < y_l).$$

□

Theorem 1.8. $(\mathbb{Q}; <) \prec (\mathbb{R}; <)$.

Proof. Both $(\mathbb{Q}; <)$ and $(\mathbb{R}; <)$ are models of DLO, DLO admits quantifier elimination, and $(\mathbb{Q}, <)$ is a substructure of $(\mathbb{R} <)$. Hence $(\mathbb{Q}, <)$ is an elementary substructure of $(\mathbb{R}, <)$. □

Example 1.9. Let $\mathcal{L} = \{+, 0\}$, $M = (\mathbb{Z}, +, 0)$ with the standard interpretations and $T = Th_{\mathcal{L}}(M)$. T does not admit quantifier elimination.