HOMEWORK 1: DUE MARCH 5, IN CLASS.

1. Homework Problems

1. Let $\mathcal{L}$ be a propositional language and φ an $\mathcal{L}$-sentence.

(i) Let $C(\varphi)$ be the total number of instance of logical connective which occur in φ, i.e. instances of $\vee, \wedge, \rightarrow \neg$.
(ii) Let $S(\varphi)$ be the total number of symbols which occur in φ.
(iii) Let $D(\varphi)$ be the total number of instance of *binary connectives* which occur in φ, i.e. instances of $\vee, \wedge, \rightarrow$.
(iv) Let $E(\varphi)$ be the total number of instances of atomic propositions which occur in φ.

E.g. if $\varphi = ((A_1 \wedge A_1) \vee (\neg A_1))$, then $S(\varphi) = 12$, $C(\varphi) = 3$, $D(\varphi) = 2$, and $E(\varphi) = 3$.

(1) Prove via structural induction that $D(\varphi) + 1 = E(\varphi)$
(2) Prove via structural induction that $S(\varphi) \geq 3C(\varphi)$.

2. Let $\mathcal{L}$ be a propositional language φ and θ be $\mathcal{L}$-formulas. We say that φ is **logically equivalent** to θ and write $\varphi \equiv \theta$ if for any $\mathcal{L}$-model M, $M \models \varphi$ if and only if $M \models \theta$.

(1) Prove (via structural induction) that any $\mathcal{L}$-sentence φ is logically equivalent to an $\mathcal{L}$-sentence θ where the only logical connectives occurring in θ are '$\neg$', '$\wedge$'.
(2) Prove (via structural induction) that any $\mathcal{L}$-sentence φ is logically equivalent to an $\mathcal{L}$-sentence θ where the only logical connectives occurring in θ are '$\neg$', '$\vee$'.

3. Let $\mathcal{L} = \{A_i : i \in \mathbb{N}\}$. Determine if the following sentences are *valid*, *satisfiable*, or *not satisfiable*. Justify your answer (a truth table is justification).

(1) $((\neg(A_1 \rightarrow A_2)) \rightarrow A_1)$.
(2) $((\neg(A_1 \rightarrow A_2)) \rightarrow A_2)$.
(3) $((A_1 \rightarrow (\neg A_2)) \rightarrow (A_1 \wedge A_2))$.
(4) $(((A_1 \vee A_2) \rightarrow (A_3 \wedge A_2)) \vee (A_2 \rightarrow A_3))$.

4. Let $\mathcal{L} = \{A_i : i \in \mathbb{N}\}$. Let $\Sigma = \{A_i \rightarrow A_{i+2} : i \in \mathbb{N}\} \cup \{A_1\}$.

(1) Find M_1 and M_2 such that $M_1 \neq M_2$ and both $M_1 \models \Sigma$ and $M_2 \models \Sigma$. Justify.
(2) Prove that Σ has no finite models.

Let $\Gamma = \{A_j \rightarrow A_{j \cdot n} : j \geq 2, n \geq 1\} \cup \{\neg A_0 \wedge \neg A_1\}$.

(1) Find M_1 and M_2 such that $M_1 \neq M_2$, both M_1 and M_2 are non-empty, and both $M_1 \models \Gamma$ and $M_2 \models \Gamma$. Justify.
(2) Prove that if $M_1 \models \Gamma$ and $M_2 \models \Gamma$, then $M_1 \cap M_2 \models \Gamma$.

5. Let E be a binary relation on the set $\{1, ..., n\}$. Let $A_{i,j}$ for $1 \leq i, j \leq n$ be distinct atomic sentences. The intended interpretation of these propositional sentences is

$$A_{i,j} := \text{“}E \text{ holds on } (i, j)\text{”}.$$

Express the following statements in propositional logic.

(1) E is a graph on $\{1, 2, ...n\}$, i.e. E is symmetric and anti-reflexive.
(2) E is a graph on $\{1, 2, ..., n\}$ and there is at least one vertex which is not adjacent to any other vertex.
(3) E is a graph on $\{1, 2, ...n\}$ and every vertex in the graph is adjacent to at least two other vertices.

In problems 6 & 7, you may not use the completeness theorem.

6. Let $\mathcal{L} = \{A_i : i \in \mathbb{N}\}$. Show that for any integer $n \geq 2$, there exists a collection of $\mathcal{L}$-sentences $\Sigma = \{\varphi_1, ..., \varphi_n\}$ such that for any proper subset $\Sigma' \subsetneq \Sigma$, Σ' is satisfiable but Σ is not satisfiable.

7. Let Σ and Γ be $\mathcal{L}$-theories.

(1) Argue that for any $\mathcal{L}$-sentences φ and ψ, $\varphi \to (\psi \to \varphi)$ is a valid sentence.
(2) Suppose that φ, ψ are $\mathcal{L}$-sentences. Prove that $\Sigma \vdash \varphi \wedge \psi$ if and only if $\Sigma \vdash \varphi$ and $\Sigma \vdash \psi$.
(3) If $\Sigma \vdash \varphi$ for every $\varphi \in \Gamma$ and $\Sigma \cup \Gamma \vdash \theta$, prove that $\Sigma \vdash \theta$.

8. We say that an $\mathcal{L}$-theory Γ is complete if for any $\mathcal{L}$-sentence φ, precisely one of the following holds: either $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg\varphi$. Prove that the following are equivalent:

(1) The deductive closure of Γ is maximally consistent, i.e. $\{\varphi : \Gamma \vdash \varphi\}$ is maximally consistent.
(2) Γ is complete.
(3) Γ has exactly one model.
(4) There is a model M such that $M \models \varphi$ if and only if $\Gamma \vdash \varphi$.

2. Extra problems

You should complete these problems, but you do not have to turn them in.

9. Prove the deduction theorem. In particular, if $\Sigma \cup \{\psi\} \vdash \varphi$, then $\Sigma \vdash \psi \to \varphi$.

10. Let $\mathcal{L} = \{A_1, ..., A_n\}$. Let $\mathcal{M}$ be the collection of $\mathcal{L}$-models. We define a map $\mu : \{\mathcal{L}\text{-sentences}\} \to [0,1]$ via $\mu(\varphi) = \frac{|\{M \in \mathcal{M} : M \models \varphi\}|}{|\mathcal{M}|}$. Intuitively, $\mu(\varphi)$ gives the probability that given a random model of $\mathcal{L}$, the sentence φ is true in M. Compute the following and prove your answers.

(1) $\mu(A_1 \wedge A_2)$.
(2) $\mu(A_1 \to A_2)$.
(3) $\mu((A_1 \to A_2) \wedge (A_2 \to A_1))$.
(4) $\mu(A_1 \wedge ... \wedge A_n)$.
(5) $\mu(A_1 \vee ... \vee A_n)$.