HOMEWORK 2: DUE MARCH 12, IN CLASS.

1. Homework Problems

Exercise 1.1. *Consider* $M := (\mathbb{R}; +, \times)$. *Write down the following formulas:*

(1) $M \models \varphi(r)$ iff $r = 0$.
(2) $M \models \varphi(r)$ iff $r = 1$.
(3) $M \models \psi(r, s)$ iff $r < s$.
(4) $M \models \varphi(r)$ iff $\sqrt{2} < r < 10.3$.
(5) $M \models \gamma(r, s, t)$ iff $(r, s) \in \mathbb{R}^2$ lie on the circle with origin $(0, 0)$, radius t.
(6) $M \models \varphi(r_1, ..., r_n)$ iff $r_1 < ... < r_n$.
(7) $M \models \phi(a, b, c, d, e, f)$ iff the points $(a, b), (c, d), (e, f) \in \mathbb{R}^2$ are non-collinear.

Exercise 1.2. *Let* $\mathcal{L} = \{+, 0\}$. *Consider*

(1) $M_1 = (\mathbb{Z}; +, 0)$, *standard interpretation.*
(2) $M_2 = (\mathbb{R}, +, 0)$, *standard interpretation.*
(3) $M_3 = (\mathbb{N}, +, 0)$, *standard interpretation.*
(4) $M_4 = (\mathbb{Z}^2, +, 0)$ *where* $(a, b) + (c, d) = (a + c, b + d)$ *and* $0 = (0, 0)$

Find $\mathcal{L}$*-sentences which do the following.*

(1) Find φ *such that* $M_1 \models \varphi$ *and* $M_2 \not\models \varphi$.
(2) Find φ *such that* $M_1 \models \varphi$ *and* $M_3 \not\models \varphi$.
(3) Find φ *such that* $M_2 \models \varphi$ *and* $M_3 \not\models \varphi$.
(4) Find φ *such that* $M_2 \models \varphi$ *and* $M_4 \not\models \varphi$.
(5) Find φ *such that* $M_1 \models \varphi$ *and* $M_4 \not\models \varphi$.

Exercise 1.3. *Consider the structures* $M_1 = (\mathbb{Z}, +)$, $M_2 = (\mathbb{Q}, \leq)$ *with the standard interpretations. Keep in mind that these are structures in separate languages.*

(1) Prove that $\mathbb{N}$ *is not a definable subset of* M_1.
(2) Prove that $\mathbb{N}$ *is not a definable subset of* M_2.

Exercise 1.4. *Let* M *and* N *be* $\mathcal{L}$*-structures.*

(1) Prove that if $M \cong N$ *then* $M \equiv N$.
(2) Prove that if $M = (A; ...)$, $N = (B; ...)$ *and* $|A|$ *is finite and* $|A| \neq |B|$, *then* $M \not\equiv N$. *Notice that we do not require* $|B|$ *to be finite.*

Exercise 1.5. *Fix* $\mathcal{L} = \{E\}$. *Let* $\mathcal{G} = (G, E)$ *be a graph. We say that two distinct points* a *and* b *are* 1*-connected if* $(a, b) \in E$. *For* $n \geq 2$, *we say that a pair of distinct points* a *and* b *in* G *are* n*-connected if there exists a path of length* n *between* a *and* b, *i.e. there exists elements* $c_1, ..., c_{n-1}$ *in* G *such that* $\{(a, c_1), (c_1, c_2), ..., (c_{n-1}, b)\} \subseteq E$.

We say that $\mathcal{G}$ *is* n*-connected if every pair of distinct points is* m*-connected for some* m *such that* $1 \leq m \leq n$.

Do the following:

(1) For each $n \geq 2$, *give an example of a graph which is* $(n + 1)$*-connected but not* n*-connected.*

(2) *Prove that for each $n \geq 2$, the collection of n-connected graphs forms an $\mathcal{L}$-elementary class.*
(3) *We say that $\mathcal{G}$ is connected if it is n-connected for some n. Prove that the collection of connected graphs does not form an elementary class.*

Exercise 1.6. *Let $\mathcal{L} = \{\leq, +, \times, 0, 1\}$. Let $M = (\mathbb{N}; \leq, +, \times, 0, 1)$ with the standard interpretations. By the extra problems, we notice that the relations 'x divides y' and 'x is prime' are definable in this language and so we let $D(x, y)$ and $T(x)$ denote these relations respectively. Let $\mathbb{P}$ be the collection of primes in $\mathbb{N}$ and $P \subseteq \mathbb{P}$.*

(1) *Prove that there exists an $\mathcal{L}$-structure $N = (A; \leq, +, \times, 0, 1)$ such that $N \equiv M$ and there exists some $a \in A$ such that for any $p \in \mathbb{P}$, $N \models D(\underbrace{1 + ... + 1}_{p-times}, a)$ if and only if $p \in P$.*
(2) *The twin prime conjecture states that there exists infinitely many pairs of primes (p_1, p_2) such that $p_2 = p_1 + 2$. Prove that if the twin prime conjecture is true, then there exists an $\mathcal{L}$-structure $N = (A; \leq, +, \times, 0, 1)$ such that $N \equiv M$ and there exists p_1 and p_2 in A with the following properties:*
   (a) $N \models T(p_1) \wedge T(p_2)$.
   (b) $N \models \underbrace{1 + ... + 1}_{n-times} < p_1$ *for each* $n \in \mathbb{N}$.
   (c) $N \models p_2 = p_1 + 1 + 1$

Exercise 1.7. *Let φ be an $\mathcal{L}$-sentence and suppose that φ is true in all infinite models of an $\mathcal{L}$-theory Σ. Show that there exists a finite number k such that φ is true in all models M of Σ which have k or more elements.*

Exercise 1.8. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be $\mathcal{L}$-elementary classes. Prove that $\mathcal{K}_1 \cup \mathcal{K}_2$ is an $\mathcal{L}$-elementary class.*

2. Extra problems

Do not turn in.

Exercise 2.1. *Consider* $M := (\mathbb{N}; +, \times)$*. Write down a formula* $\varphi(x)$ *such that*

1. $M \models \varphi(n)$ *iff* $n = 0$.
2. $M \models \varphi(n)$ *iff* $n = 1$.
3. $M \models \varphi(n)$ *iff* n *is prime.*
4. $M \models \varphi(n)$ *iff* n *is the sum of two primes.*
5. $M \models \varphi(n)$ *iff* n *is a perfect square.*
6. $M \models \varphi(n)$ *iff* n *is a perfect number.*
7. $M \models \varphi(n)$ *iff* $n = 37$ *or* $n = 100$.

Write down a formula $\psi(x, y)$ *in the language* $\mathcal{L} = \{+, \times\}$ *such that*

1. $M \models \psi(n, m)$ *iff* $n < m$.
2. $M \models \psi(n, m)$ *iff* $n = m + 1$.
3. $M \models \psi(n, m)$ *iff* n *divides* m.
4. $M \models \psi(n, m)$ *iff* n *and* m *are consecutive primes.*
5. $M \models \psi(n, m)$ *iff* $n - m$ *is composite.*

Exercise 2.2. *Consider* $M := (\mathbb{R}; +, \times)$*. Write down the following formulas:*

1. $M \models \phi(a, b, c, d, e, f, g, h)$ iff the points $(a, b), (c, d), (e, f) \in \mathbb{R}^2$ are non-collinear and the point (g, h) lies in the interior of the triangle defined by the vertices $(a, b), (c, d), (e, f)$.