HOMEWORK 3: DUE MARCH 19TH, IN CLASS.

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Try to prove the following fact on your own. You can use it in your computations throughout this assignment.

Fact 0.1. *Let κ be an infinite cardinal. Consider $\{A_i : i \in I\}$ where $|I| \leq \kappa$ and for each $i \in I$, $|A_i| \leq \kappa$. Then $|\bigcup_{i \in I} A_i| \leq \kappa$. Moreover,*

1. *if $A_i = \kappa$ for some $i \in I$, then $|\bigcup_{i \in I} A_i| = \kappa$.*
2. *If $|I| = \kappa$ and each $i \in I$, $A_i \neq \emptyset$, then $|\bigcup_{i \in I} A_i| = \kappa$.*

1. Homework problems

Exercise 1.1. *Let $(A, <)$ be a total ordering. Prove the following are equivalent:*

1. *Every non-empty subset of A has a least element.*
2. *$(A, <)$ has no infinite descending chain.*

Exercise 1.2. *Let C be any set. Using Zorn's lemma, prove that C can be well-ordered, i.e. there exists an ordering $<$ such that $(C, <)$ is well-ordered.*

Exercise 1.3. *Let A, B be sets. If there exists $f : A \to B$ and $g : B \to A$ which are injections, prove that there exists a bijection $h : A \to B$.*

Exercise 1.4. *When κ and λ are cardinals, the notation κ^λ is the cardinality of the set of all functions from λ to κ, i.e. $\kappa^\lambda = |\{f | f : \lambda \to \kappa\}|$. Prove the following.*

(a) If A is a set, then $|\mathcal{P}(A)| = 2^{|A|}$.
(b) (Do not turn in) For any infinite set A, $2^{|A|} = 3^{|A|}$.
(c) $|\mathbb{R}| = 2^{\aleph_0}$. $\aleph_0$ is just a fancy name for ω or $|\mathbb{N}|$.

Exercise 1.5. *Let $\mathcal{L} = \{E\}$ where E is a binary relation. Let T_E be the first order theory consisting of the following sentences.*

1. *E is an equivalence relation.*
2. *E has infinitely many equivalence classes.*
3. *E each equivalence class has infinitely many elements.*

Write T_E as a collection of first order sentences.

Exercise 1.6. *Consider T_E from above. Compute the following:*

(a) $I(T_E, \aleph_0)$.
(b) $I(T_E, \aleph_n)$ for $n \in \mathbb{N}$.
(c) $I(T_E, \aleph_\omega)$.

Exercise 1.7. *Give an example of a complete theory T (with no finite models) which is not $\aleph_0$-categorical but is $\aleph_1$-categorical. Prove your claim.*

Exercise 1.8. *Determine if the following structures are countably categorical. Justify.*

(1) The theory of $(\mathbb{Z}; S)$ *where* S *is the successor function.*
(2) The theory of $(\mathbb{N}; \leq)$.
(3) The theory T *which is constructed as follows: Let* $\mathcal{L} = \{P_i : i \in \mathbb{N}/\{0\}\}$ *where each* P_i *is a unary relation symbol. Suppose that* T *says that*
   (a) Each P_i *is infinite.*
   (b) For each $i \neq j$, P_j *and* P_i *are disjoint.*