**HOMEWORK 4 - DUE MARCH 26, IN CLASS.**

**Exercise 0.1.** *Consider the language* $\mathcal{L} = \{\leq, (c_i)_{i\in\mathbb{N}}\}$. *Let* $T = T_{\leq} \cup \{c_i < c_j : i < j\}$ *where* $T_{\leq}$ *is the theory of dense linear orderings without endpoints. Prove that* $I(T, \aleph_0) \geq 3$.

**Exercise 0.2.** *Let* $\mathcal{L} = \{R\}$ *where* $R$ *is a binary relation symbol. Let* $T_R$ *be the theory of the random graph, i.e.* $T_R$ *is the theory of a graph along with the family of sentences, for* $n, m \geq 0$,

$$\forall x_1, ..., x_n \forall y_1, ...y_m(\{x_1, ..., x_n\} \cap \{y_1, ..., y_m\} = \emptyset$$
$$\to \exists z(\bigwedge_{i=1}^{n} z \neq x_i \wedge \bigwedge_{i=1}^{m} z \neq y_i \wedge \bigwedge_{i=1}^{n} R(z, x_i) \wedge \bigwedge_{i=1}^{m} \neg R(z, y_i))).$$

*Prove that* $T_R$ *is consistent. Hint: Build a model* $T_R$ *in countably many stages.*

**Exercise 0.3.** *Prove that if* $M \models T_R$, *and if* $a_1, ..., a_n$ *and* $b_1, ..., b_m$ *are in* $M$ *such that* $\{a_1, ..., a_n\} \cap \{b_1, ..., b_m\} = \emptyset$, *then there exists and infinite set* $D \subset M$ *such that for any* $d \in D$, $M \models \bigwedge_{i=1}^{n} R(a_i, d) \wedge \bigwedge_{i=1}^{m} \neg R(b_i, d)$.

**Exercise 0.4.** *Prove that* $T_R$ *is countably categorical. Conclude that* $T_R$ *is complete. Hint: Use the back-and-forth method (similar to DLO).*

**Exercise 0.5.** *Let* $M$ *be an* $\mathcal{L}$*-structure and* $B \subseteq M$. *Let* $\pi$ *be a type in variable* $x$ *over* $B$. *Prove that there exists a complete type* $p$ *(i.e.* $p \in S_x(B)$*) such that* $\pi \subseteq p$.

**Exercise 0.6.** *Let* $M$ *be an* $\mathcal{L}$*-structure and fix a variable* $x$. *Prove that there exists a model* $N$ *such that*

*(1)* $M \preceq N$.
*(2) For any* $p \in S_x(M)$, *there exists some* $b \in N$ *such that* $b \models p$.

**Exercise 0.7.** *Let* $(\omega_1, \in)$ *be the first uncountable ordinal. Prove that there exists a countable ordinal* $\alpha$ *such that* $(\alpha, \in) \prec (\omega_1, \in)$. *Hint: Think about the proof of downward Lowenheim-Skolem.*

**Exercise 0.8.** *Student's choice: Only prove one of the following (one is harder than the other):*

*(1) Prove that if* $N \preceq M$ *and* $M \preceq K$, *then* $N \preceq K$.
*(2) Determine (and provide a proof of your decision) if* $(\{(r, 0) : r \in \mathbb{R}\}; +, 0) \prec (\mathbb{R}^2; +, 0)$.