HOMEWORK 5 - DUE APRIL 2ND, IN CLASS.

Exercise 0.1. *Prove that* $I(DLO, \aleph_1) \geq 2^{\aleph_0}$.

Exercise 0.2. *Prove that the evens are not a definable subset of* $(\mathbb{N}, S)$. *Hint: Suppose the evens are definable. Construct automorphism in an elementary extension to get a contradiction.*

Exercise 0.3. *Let* I *be finite and suppose that* D *is an ultrafilter on* I. *Prove that* D *is principle, i.e. there exists some* $i \in I$ *such that* $D = \{X \subseteq I : i \in X\}$.

Definition 0.4. Let M and N be $\mathcal{L}$-structures. We say that $G : M \to N$ is an elementary embedding if

(1) G is injective.
(2) For any formula $\varphi(x_1, ..., x_n)$, we have

$$M \models \varphi(a_1, ..., a_n) \iff N \models \varphi(G(a_1), ..., G(a_n)).$$

Exercise 0.5. *Let* $I = \omega$ *and* D *be an ultrafilter on* I. *Let* M *be an* $\mathcal{L}$*-structure and consider* $(M_i)_{i \in I}$ *where for each* $i \in I$, $M_i = M$. *Then* $\prod_D M_i$ *is called an ultrapower. Let* $\Delta : M \to \prod_D M_i$ *via* $\Delta(a) = [f_a]_D$ *where* $f_a(i) = a$ *for every* $i \in I$. *Prove that this map is an elementary embedding.* Δ *is called the diagonal map. Hint: Don't use induction.*

Exercise 0.6. *Let* I *be an infinite indexing set,* $(M_i)_{i \in I}$ *a sequence of* $\mathcal{L}$*-structures, and* D *be an ultrafilter on* I. *Determine if the following statements are true or false. Justify your claims:*

(1) If $\mathcal{L}$ *is the language of graphs and* M_i *are connected graphs (i.e. between any two points, there exists a path), then* $\prod_D M_i$ *is connected.*
(2) If D *is principle, then there exists some* $i \in I$ *such that* $M_i \cong \prod_D M_i$.
(3) If D_1 *and* D_2 *are non-principle ultrafilters on* I *then* $\prod_{D_1} M_i \equiv \prod_{D_2} M_i$.

Definition 0.7. We say that an $\mathcal{L}$-structure M is pseudofinite if there exists an indexing set I, and ultrafilter D, and a family of **finite** $\mathcal{L}$-structures $(M_i)_{i \in I}$ such that $\prod_D M_i \equiv M$. We say that a complete theory

Exercise 0.8. *Determine if the following are pseudofinite. Justify your claims.*

(1) $(\mathbb{Q}; \leq)$.
(2) $(\mathbb{N}; f)$ *where* $f(n) = \lceil n/2 \rceil$.
(3) The theory of infinitely many equivalence classes all with infinitely many elements (complete by Vaught's test).

Exercise 0.9. *Do not need to write the proofs; Determine if the following admit quantifier elimination:*

(1) The theory of $(\mathbb{N}; S)$.
(2) The theory of $(\mathbb{N}; S, 0)$.
(3) The theory of $(\mathbb{N}; \leq)$.
(4) The theory of $(\mathbb{N}; \leq, 0)$.

Exercise 0.10. *Let $\mathcal{L} = \{<\}$, $I = \omega$, and M_i be a collection of total orderings* ***which are not well-ordered****, and D be a non-principle ultrafilter on I. Is it necessarily true that $\prod_D M_i$ is not well-ordered? Prove you claim.*