HOMEWORK 8 - DUE MAY 21ST, IN CLASS.

Exercise 0.1. *Do not turn in: Prove that ACF_0 admits quantifier elimination.*

Exercise 0.2. *Prove that ACF_0 is strongly minimal.*

Exercise 0.3. *Suppose that $RM(\psi(x)) = \infty$. Prove that there exists $\psi_1(x)$ and $\psi_2(x)$ such that*

(1) $\psi_1(x) \subseteq \psi(x)$, $\psi_2(x) \subseteq \psi(x)$.
(2) $\psi_1(x) \cap \psi_2(x) = \emptyset$.
(3) $RM(\psi_1(x)) = RM(\psi_2(x)) = \infty$.

Definition 0.4. Let $\varphi(x, y)$ be a formula in $\mathcal{L}_{xy}(M)$. We say that $\varphi(x, y)$ is "k-stable relative to M" if there does not exists finite sequences $(a_i)_{1\leq i\leq k}$ and $(b_j)_{1\leq j\leq k}$ such that

$$M \models \varphi(a_i, b_j) \iff i \leq j.$$

Exercise 0.5. *Let $M \prec \mathcal{U}$ and $\varphi(x, y) \in \mathcal{L}_{xy}(M)$ and $\mathcal{U}$ is a monster model. Prove the following:*

(1) *If $\varphi(x, y)$ is k-stable relative to M, then $\varphi(x, y)$ is k-stable relative to $\mathcal{U}$.*
(2) *If $\varphi(x, y) \in \mathcal{L}_{xy}(\emptyset)$, and $\varphi(x, y)$ is k-stable to some model of M, then $\varphi(x, y)$ is k-stable relative to any model of $Th_{\mathcal{L}}(M)$.*
(3) *If $\varphi(x, y)$ is not stable relative to M, then there exists $(a_i)_{i<\omega}$ and $(b_j)_{j<\omega}$ in $\mathcal{U}$ such that*

$$\mathcal{U} \models \varphi(a_i, b_j) \iff i < j.$$

Example 0.6. Determine whether the following formulas are stable. If yes, find the smallest k such that the formula is k-stable.

(1) $\varphi(x, y) := \exists z(x \cdot z = y)$ in $(\mathbb{Z}; \cdot)$.
(2) $\varphi(x, y) := \exists z(x \cdot z = y)$ in $(\mathbb{R}; \cdot)$.
(3) $\varphi(x, y) := \exists z(x - y = z \cdot z)$ in $(\mathbb{R}; +, \cdot, -)$.
(4) $\varphi(x, y) := \exists z(x - y = z \cdot z)$ in $(\mathbb{C}; +, \cdot, -)$.

Exercise 0.7. *Let $[\mathbb{Z}]^n$ be the collection of unordered subsets of $\mathbb{Z}$ with precisely n elements. Consider the relation R where for $A_1, A_2 \in [\mathbb{Z}]^n$, we have that $[\mathbb{Z}]^n \models R(A_1, A_2)$ if and only if $A_1 \cap A_2 \neq \emptyset$. Is the formula $R(x, y)$ stable?*

What about the same formula defined in the structure $[\mathbb{Z}]^\omega$?

Exercise 0.8. *Consider $X = [0, 1]$ as a topological space. Let $Hom(X)$ be the collection of homeomorphisms from X to itself. Find a formula in the language $\mathcal{L}_{grp}$ which is unstable.*

Exercise 0.9. *Let K be any field and consider the formula*

$$\varphi(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) := det\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}\right) \neq 0.$$

Argue that

(1) *The formula is definable.*
(2) *The formula is stable.*