MEASURES AND STABILITY IN A MODEL

K. GANNON

Abstract. We prove that if a formula is *stable in a model*, then every local Keisler measure on the associated local type space is a convex combination of (at most countably many) types. Using this, we give an elementary proof of "Fubini's theorem" in this context.

1. Introduction

We prove what is stated in the abstract. We begin with some notation. Let x, y be tuples and let $\varphi(x; y)$ be a partitioned formula in a language $\mathcal{L}$ with variables x and parameters y. Let $\varphi^*(y; x)$ be the same formula as $\varphi(x; y)$, but with exchanged roles for the variables and parameters. We recall the definition of *stable in a model*:

Definition 1.1. A formula $\varphi(x; y)$ is *stable in an $\mathcal{L}$-structure M* if for any two sequences $(a_n)_{n\in\mathbb{N}}, (b_m)_{m\in\mathbb{N}}$ where $a_n \in M^x$ and $b_m \in M^y$, we have that

$$\lim_m \lim_n \varphi(a_n, b_m) = \lim_n \lim_m \varphi(a_n, b_m),$$

provided both limits exist, where $\varphi(a_n, b_m) = \begin{cases} 1 & M \models \varphi(a_n, b_m), \\ 0 & \text{otherwise.} \end{cases}$

Let $S_\varphi(M)$ be the space of φ-types with parameters from M. Let $\mathbb{B}_\varphi(M)$ be the Boolean algebra of definable subsets of M generated by $\{\varphi(x, b) : b \in M\}$. We will routinely identify definable sets with the formulas which define them. A φ-formula is an element of $\mathbb{B}_\varphi(M)$. Likewise, we have analogous definitions for $S_{\varphi^*}(M)$ and $\mathbb{B}_{\varphi^*}(M)$. A φ^*-definition for a type p in $S_\varphi(M)$ is a φ^*-formula, $d^p_{\varphi^*}(y)$, such that for each $b \in M^y$, $\varphi(x, b) \in p$ if and only if $M \models d^p_{\varphi^*}(b)$. Finally, we let $\mathfrak{M}_\varphi(M)$ and $\mathfrak{M}_{\varphi^*}(M)$ denote the spaces of finitely additive probability measures on $\mathbb{B}_\varphi(M)$ and $\mathbb{B}_{\varphi^*}(M)$ respectively. We recall that we can identify a measure in each of these spaces canonically with a regular Borel probability measure on their corresponding type space, e.g. $\mathfrak{M}_\varphi(M)$ is in canonical correspondence with regular Borel probability measures on $S_\varphi(M)$.

In [1], Ben Yaacov established a surprising connection between functional analysis and local stability. In particular, he gave a proof of the *fundamental theorem of stability* using Grothendieck's double limit theorem [2]. Via the double limit theorem, he showed:

Theorem 1.2. *Assume that $\varphi(x; y)$ is stable in M, $p \in S_\varphi(M)$, and $q \in S_{\varphi^*}(M)$. Then p has a φ^*-definition $d^p_{\varphi^*}(y)$, q has a φ-definition $d^q_\varphi(x)$, and $d^p_{\varphi^*}(y) \in q$ if and only if $d^q_\varphi(x) \in p$.*

It is natural to ask "What do Keisler measures looks like in this context?". We will show that finitely additive probability measures are simply "sums of types". Recall that Keisler showed in [3] that if a formula $\varphi(x; y)$ is k-stable for some k, i.e. there do not exist $a_1, ..., a_k$,

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$b_1,...,b_k$ so that $M \models \varphi(a_i, b_j)$ if and only if $i < j$, then every finitely additive probability measure on $\mathbb{B}_\varphi(M)$ is at most a countable sum of "weighted" types. From Theorem 1.2 and an application of the Sobczyk-Hammer Decomposition Theorem, we prove the following,

Theorem 1.3. *Let $\varphi(x; y)$ be stable in M and assume that $\mu \in \mathfrak{M}_\varphi(M)$. Then $\mu = \sum_{i \in I} r_i \delta_{p_i}$ where I is some initial segment of $\mathbb{N}$*[1]*, each p_i is in $S_\varphi(M)$, δ_{p_i} is the corresponding Dirac measure at p_i, each r_i is a positive real number (strictly greater than 0), and $\sum_{i \in I} r_i = 1$.*

To be clear, a formula φ is stable in M if and only if φ^* is stable in M. Therefore, Theorem 1.3 can also be applied to $\mathfrak{M}_{\varphi^*}(M)$. We note that from this description of measures in this context, we have almost for free the following corollary,

Corollary 1.4 (Local Fubini)**.** *Assume that $\varphi(x; y)$ is stable in M. Let $\mu \in \mathfrak{M}_\varphi(M)$ and $\nu \in \mathfrak{M}_\varphi(M)$. Let $F_\mu^\varphi : S_{\varphi^*}(M) \to \mathbb{R}$ via $F_\mu^\varphi(q) = \mu(d_\varphi^q(x))$. Let $F_\nu^{\varphi^*} : S_\varphi(M) \to \mathbb{R}$ via $\nu(d_{\varphi^*}^p(y))$. Then the maps F_μ^φ and $F_\nu^{\varphi^*}$ are measurable and*

$$\int_{S_\varphi(M)} F_\nu^{\varphi^*}(p) d\mu = \int_{S_{\varphi^*}(M)} F_\mu^\varphi(q) d\nu,$$

where we have identified μ and ν with their corresponding regular Borel measures on $S_\varphi(M)$ and $S_{\varphi^*}(M)$ respectively.

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2. Local Measures and Stability in a Model

The proof of Theorem 1.3 uses the Sobczyk-Hammer decomposition theorem for positive, bounded charges. We state the theorem for finitely additive probability measures. Before referencing this theorem, we establish a convention and recall two kinds of measures.

Remark 2.1. We will say that $\mathbb{B}$ is a Boolean algebra on X if $\mathbb{B} \subset \mathcal{P}(X)$ and $\mathbb{B}$ is a Boolean algebra under the standard interpretation of of union, intersection, complement, etc. We also remark that X and $\emptyset$ are elements of $\mathbb{B}$.

Definition 2.2. Let $\mathbb{B}$ be a Boolean algebra on a set X and μ be a finitely additive probability measure on $\mathbb{B}$.

1. We say that μ is *strongly continuous* on $\mathbb{B}$ if for all $\epsilon > 0$ there exist $F_1, ..., F_n \in \mathbb{B}$ such that $\{F_i\}_{i=1}^n$ form a partition of X and for each i, $\mu(F_i) < \epsilon$.
2. We say that μ is *0-1 valued* on $\mathbb{B}$ if for every F in $\mathbb{B}$, $\mu(F) = 0$ or $\mu(F) = 1$.

We refer the reader to [5, Theorem 5.2.7] for a proof of the following theorem.

Theorem 2.3 (Sobczyk-Hammer Decomposition Theorem [4])**.** *Let $\mathbb{B}$ be a Boolean algebra on X and μ be a finitely additive probability measure on $\mathbb{B}$. Then, there exists an (not necessarily proper) initial segment I of $\mathbb{N}$, a sequence of distinct finitely additive probability measures $(\mu_i)_{i \in I}$, and a sequence of positive real numbers $(r_i)_{i \in I}$ where each $r_i \geq 0$, with the following properties,*

(i) μ_0 is strongly continuous on $\mathbb{B}$,
(ii) μ_i is 0-1 valued on $\mathbb{B}$ for every $i \geq 1$,

[1] I need not be a *proper* initial segment. $I = \{0, ..., n\}$ for some n or $I = \mathbb{N}$

(iii) $\sum_{i\in I} r_i = 1$, *and*
(iv) $\mu = \sum_{i\in I} r_i\mu_i$.

Further, the decomposition in (iv) is unique (up to permutation of the sequence).

The Sobczyk-Hammer decomposition theorem allows us to decompose any finitely additive probability measure into a single strongly continuous measure and a convex combination of (at most countably many) 0-1 valued measures. We will show that if $\varphi(x;y)$ is stable in M, then there do not exist any strongly continuous measures on $\mathbb{B}_\varphi(M)$. Therefore, every finitely additive probability measure will be the "weighted sum" of at most countably many types.

2.1. **Proof of Theorem 1.3.**

Definition 2.4 (2-Tree)**.** Let $\mathbb{B}$ be a Boolean algebra on a set X. We say that $\mathbb{B}$ has a *2-tree* if there exists $T \in \mathcal{P}(\mathbb{B})$ such that $(T, \supsetneq)$ is an infinite, complete, binary tree, and if $A, C \in T$, $A \not\supset C$, and $C \not\supset A$, then $A \cap C = \emptyset$.

Fact 2.5. *Let $\mathbb{B}$ be a Boolean algebra on a set X and assume that $\mathbb{B}$ has a 2-tree. Then $|Ult(\mathbb{B})| \geq 2^{\aleph_0}$ where $Ult(\mathbb{B})$ is the set of ultrafilters on $\mathbb{B}$.*

Proof. Let γ be a path in T and let $A_\gamma = \{B \in T : B \in \gamma\}$. Clearly, A_γ has the finite intersection property (since if $B, C \in A_\gamma$, then either $B \subset C$ or $C \subset B$). Then, A_γ can be extended to an ultrafilter over $\mathbb{B}$. For each path γ, let U_γ be an ultrafilter extending A_γ. Now, assume that δ, γ are two different paths in T. Assume that $U_\gamma = U_\delta = U$. Since γ, δ are two separate paths, there exists $A \in \gamma$ and $B \in \delta$ such that $A \not\subset B$ and $B \not\subset A$. Then $A \cap B = \emptyset$ and therefore U cannot extend both A_γ and A_δ. Therefore, we have at least $2^{\aleph_0}$ many ultrafilters on $\mathbb{B}$. □

Lemma 2.6. *Let $\mathbb{B}$ be a Boolean algebra on a set X. Assume that there exists a strongly continuous measure μ over $\mathbb{B}$. Then $\mathbb{B}$ has a 2-tree.*

Proof. Using μ, we will build a 2-tree. We build this tree in steps:
Stage 0: Let $T_0 = \{X\}$.
Stage $n+1$: We construct a tree of height $n+1$. Assume that T_n is a (complete) binary tree of height n such that for each $A \in T_n$, $\mu(A) > 0$. Assume furthermore that if $A, B \in T$ and $A \not\supset B$ and $B \not\supset A$, then $A \cap B = \emptyset$. We will construct T_{n+1} by adding two children to each leaf. Let $\mathbb{L}_n$ be the collection of leaves on T_n. By assumption, each node of our tree has positive measure, therefore for each $L \in \mathbb{L}_n$, $\mu(L) > 0$. Let $\epsilon = \frac{\min\{\mu(L) : L \in \mathbb{L}\}}{2}$. Now, since μ is strongly continuous, there exist $H_1, ..., H_m \in \mathbb{B}$ such that $\mathbb{H} = \{H_1, ..., H_m\}$ partitions X and $\mu(H) < \epsilon$ for each $H \in \mathbb{H}$. Now fix a leaf L_i. Consider $L_i \cap \mathbb{H} = \{L_i \cap H_j : H_j \in \mathbb{H}\}$. We notice that $L_i \cap \mathbb{H}$ forms a partition of L_i. Therefore, we have that

$$0 < \mu(L_i) = \mu\Big(\bigcup_{K \in L_i \cap \mathbb{H}} K\Big) = \sum_{K \in L_i \cap \mathbb{H}} \mu(K).$$

Hence, there exists $K_r \in L_i \cap \mathbb{H}$ such that $\mu(K_r) > 0$. Furthermore, we note that

$$\mu(K_r) = \mu(L_i \cap H_r) \leq \mu(H_r) < \epsilon \leq \frac{L_i}{2}.$$

By the above, we note that $\mu(K_r) < \mu(L_i)$. Therefore there must exist some $K_l \in L_i \cap \mathbb{H}$ such that $K_l \neq K_r$ and $\mu(K_l) > 0$. We now add K_r, K_l as children for L_i. Let T_{n+1} be the tree constructed after repeating this process for each $L \in \mathbb{L}_n$. Clearly, T_{n+1} is a binary tree of height $n+1$ such that for each $A \in T_{n+1}$, $\mu(A) > 0$.

Now let $T = \bigcup_{n \geq 0} T_n$. T is clearly a 2-tree by construction. □

Definition 2.7. Let $Red_\varphi(M)$ be the reduct of M to language $L_\varphi = \{\varphi\}$. Then, we say that a subset N of M is a φ-substructure of M,written $N \prec_\varphi M$, if $Red_\varphi(N) \prec Red_\varphi(M)$.

Theorem 2.8. *Assume that $\varphi(x;y)$ is stable in M. Then there are no strongly continuous measures on $\mathbb{B}_\varphi(M)$.*

Proof. Assume that there exists a strongly continuous measure over $\mathbb{B}_\varphi(M)$. By the Lemma 2.6 and Fact 2.5, we know that there exists a countable subalgebra $\mathbb{B}_0 \subset \mathbb{B}_\varphi(M)$ such that $Ult(\mathbb{B}_0) \geq 2^{\aleph_0}$ (i.e. $\mathbb{B}_0$ is generated by the collection of subsets of M which appear in our infinite binary tree). Choose $C \subset M$ such that for each $B \in \mathbb{B}_0$, there exists $b_1, ..., b_n$ in C such that B is an element of the boolean algebra generated by $\{\varphi(x; b_i) : i \leq n\}$. Notice that since $\mathbb{B}_0$ is countable, we can choose C to be countable. By the Downward Löwenheim-Skolem theorem, there exists $N \prec_\varphi M$ such that $C \subset N$ and $|N| = \aleph_0$. Then,

$$2^{\aleph_0} \leq |Ult(\mathbb{B}_\varphi(C))| \leq |Ult(\mathbb{B}_\varphi(N))| = |S_\varphi(N)|.$$

However, by stability, every φ-type over N is definable by a φ^*-formula with parameters from N. Since $|N| = \aleph_0$, there are only countably many φ^*-formulas. Therefore, not every φ-type is definable. Hence, $\varphi(x;y)$ is unstable in N. Since $N \prec_\varphi M$, by definition we have $N \subset M$ and so $\varphi(x;y)$ is unstable in M. □

Corollary 2.9. *Let $\varphi(x;y)$ be stable in M and let μ be a finitely additive probability measure on $\mathbb{B}_\varphi(M)$. Then there exists an (not necessarily proper) initial segment I of $\mathbb{N}$ such that $\mu = \sum_{i \in I} r_i \delta_{p_i}$ where $p_i \in S_\varphi(M)$, $\sum_{i \in I} r_i = 1$, and each $r_i > 0$.*

Proof. By the Sobczyk-Hammer Decomposition Theorem, any finitely additive measure on $\mathbb{B}_\varphi$ is the a convex combination of a strongly continuous measure and (at most) countably many {0-1} valued measures. Since there are no strongly continuous measures on $\mathbb{B}_\varphi$, every measure is the "weighted" sum of at most countably 0-1 valued measures. Every 0-1 valued measure is of the form δ_p for some $p \in S_\varphi(M)$, which completes the proof. □

2.2. **Proof of Corollary 1.4.** In this subsection, we prove the local version of Fubini's theorem.

Proposition 2.10. *Assume that $\varphi(x;y)$ is stable in M. Then the maps $F_\mu^\varphi, F_\nu^{\varphi^*}$ as defined in Corollary 1.4 are well defined and measurable. In particular, they are continuous.*

Proof. By symmetry, we only need to show the proposition for F_μ^φ. By Theorem 1.3, $\mu = \sum_{i \in I} r_i \delta_{p_i}$. Since every type is definable, we know that for each $p \in S_\varphi(M)$, the map $F_{\delta_p}^\varphi : S_{\varphi^*}(M) \to \mathbb{R}$ is continuous. Notice that $F_\mu^\varphi = \sum_{i \in I} r_i F_{\delta_{p_i}}$. If $I = \{0, ..., n\}$, then F_μ^φ is clearly continuous. If $I = \mathbb{N}$, let $g_N = \sum_{i=1}^N r_i F_{\delta_{p_i}}^\varphi$. Then, each g_N is continuous and the sequence $(g_N)_{N \in \mathbb{N}}$ converges uniformly to F_μ^φ, so F_μ^φ is continuous. □

Proposition 2.11. *Assume that $\varphi(x;y)$ is stable in M, $p \in S_\varphi(M)$, and $\nu \in \mathfrak{M}_{\varphi^*}(M)$. Then,*

$$\int_{S_{\varphi^*}(M)} F_{\delta_p}^\varphi d\nu = \int_{S_\varphi(M)} F_\nu^{\varphi^*} d\delta_p.$$

Proof. We compute both terms. First, we compute the LHS.

$$\int_{S_\varphi(M)} F_\nu^{\varphi^*} d\delta_p = F_\nu^{\varphi^*}(p) = \nu(d_{\varphi^*}^p(y)).$$

Now the RHS. Using Theorem 1.2, we compute

$$\int_{S_{\varphi^*}(M)} F_{\delta_p}^{\varphi} d\nu = \nu\Big(\big\{q \in S_{\varphi^*}(M) : F_{\delta_q}^{\varphi}(q) = 1\big\}\Big) = \nu\Big(\big\{q \in S_{\varphi^*}(M) : \delta_q(d_\varphi^q(x)) = 1\big\}\Big)$$
$$= \nu\Big(\big\{q \in S_{\varphi^*}(M) : d_\varphi^q(x) \in p\big\}\Big) = \nu\Big(\big\{q \in S_{\varphi^*}(M) : d_{\varphi^*}^p(y) \in q\big\}\Big) = \nu(d_{\varphi^*}^p(y)).$$

□

Theorem 2.12. *Assume that* $\varphi(x;y)$ *is stable in* M*. Let* $\mu \in \mathfrak{M}_\varphi(M)$ *and* $\nu \in \mathfrak{M}_{\varphi^*}(M)$*. Then*

$$\int_{S_{\varphi^*}(M)} F_\mu^{\varphi} d\mu = \int_{S_\varphi(M)} F_\nu^{\varphi^*} d\mu.$$

Proof. By stability in M, $\mu = \sum_{i \in I} r_i \delta_{p_i}$. Then we compute

$$\int_{S_{\varphi^*}(M)} F_\mu^{\varphi} d\nu = \lim_{N\to\infty} \int_{S_{\varphi^*}(M)} \sum_{i=1}^N r_i F_{\delta_{p_i}}^{\varphi} d\nu = \lim_{N\to\infty} \sum_{i=1}^N r_i \int_{S_\varphi(M)} F_\nu^{\varphi^*} d\delta_{p_i}$$
$$= \lim_{N\to\infty} \int_{S_\varphi(M)} F_\nu^{\varphi^*} d\Big(\sum_{i=1}^N r_i \delta_{p_i}\Big) = \int_{S_\varphi(M)} F_\nu^{\varphi^*} d\mu.$$

The computations above are all straight forward to verify. We now give a few overkill justifications. The first equality follows from the dominated convergence theorem. The third equality follows from Proposition 2.11 and linearity of integration. The last equality follows from the measures $\sum_{i=1}^N r_i \delta_{p_i}$ converging in (the total variation) norm to μ. □

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Department of Mathematics, University of Notre Dame, Notre Dame, IN, 46656, USA
E-mail address: kgannon1@nd.edu