Gaussian deconvolution and the lace expansion

Yucheng Liu

Joint work with Gordon Slade

University of British Columbia

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Convolution equations on $\mathbb{Z}^d$

- $f * g(x) = \sum_{y \in \mathbb{Z}^d} f(y) g(x-y)$
- Random walk two-point function ($d > 2$):
  Let $D(x) = \frac{1}{2d} \mathbb{1}\{|x| = 1\}$ and $\delta(x) = \delta_{0,x} = \mathbb{1}\{x = 0\}$, then

$$C(x) = \sum_{n=0}^{\infty} D^{*n}(x) = \delta_{0,x} + D(x) + D * D(x) + \cdots$$

  satisfies the convolution equation $C = \delta + D * C$.
- (Bond) Percolation connection probability (d large):
  Let $\tau_p(x) = \mathbb{P}_p(0 \leftrightarrow x)$, then for $p \le p_c$,

$$\tau_p = \delta + \Pi_p + pD * (\delta + \Pi_p) * \tau_p.$$

- Self-avoiding walk two-point function ($d > 4$): For $z \le z_c$,

$$G_z = \delta + zD * G_z + \Pi_z * G_z.$$

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Deconvolution

- Random walk: Since $C(x) = \sum_{n=0}^{\infty} D^{*n}(x)$ satisfies $C = \delta + D * C$, we can rearrange it into

$$(\delta - D) * C = \delta.$$

So C is the *deconvolution* of the operator $\delta - D$, which is minus the discrete Laplacian. We also call $C(x)$ the *lattice Green function*. It is well-known that as $|x| \to \infty$,

$$C(x) = \frac{a_d}{|x|^{d-2}} + O\left(\frac{1}{|x|^d}\right), \qquad a_d = \frac{d\Gamma(\frac{d-2}{2})}{2\pi^{d/2}}.$$

- We consider the convolution equation

$$F * G = \delta$$

with a given $F : \mathbb{Z}^d \to \mathbb{R}$ and prove $G(x) \sim \text{const} \cdot |x|^{-(d-2)}$ under some assumptions on F.

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Fourier transform

- Let $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ be the continuum torus, which we identify with $(-\pi, \pi]^d \subset \mathbb{R}^d$. We will use the L^1 Fourier transform

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{ik\cdot x} \qquad (k \in \mathbb{T}^d)$$

and the inverse Fourier transform

$$f(x) = \int_{\mathbb{T}^d} \hat{f}(k) e^{-ik\cdot x} \frac{dk}{(2\pi)^d} \qquad (x \in \mathbb{Z}^d).$$

- We will also use the L^2 Fourier transform.
- Random walk example:

$$C(x) = \int_{\mathbb{T}^d} \frac{e^{-ik\cdot x}}{1 - \hat{D}(k)} \frac{dk}{(2\pi)^d}, \qquad \hat{D}(k) = d^{-1} \sum_{j=1}^{d} \cos k_j$$

(cf. $(\delta - D) * C = \delta$).

Main result: Gaussian deconvolution

We solve $F * G = \delta$ using Fourier integral $G(x) = \int_{\mathbb{T}^d} \frac{e^{-ik\cdot x}}{\hat{F}(k)} \frac{dk}{(2\pi)^d}$.

Theorem (Hara'08, L.–Slade'23)

Let $d > 2$. Suppose $F : \mathbb{Z}^d \to \mathbb{R}$ is a $\mathbb{Z}^d$-symmetric function, and suppose there are $K_1, K_2 > 0$, $\rho > \max(0, \frac{d-8}{2})$ such that, for all $x \in \mathbb{Z}^d$ and $k \in \mathbb{T}^d$,

$$|F(x)| \le \frac{K_1}{|x|^{d+2+\rho}}, \qquad \hat{F}(0) = 0, \qquad \hat{F}(k) - \hat{F}(0) \ge K_2 |k|^2.$$

Then

$$G(x) \sim \frac{a_d}{\kappa |x|^{d-2}} \qquad \text{as } |x| \to \infty,$$

where $\kappa = -\sum_{x\in\mathbb{Z}^d} |x|^2 F(x) \in (0,\infty)$.

We do not assume $F(x) \le 0$ for $x \ne 0$.

Gaussian deconvolution

- The theorem was first proved by Hara in 2008 using intricate Fourier analysis, without the assumption that $\rho > \frac{d-8}{2}$ (only requiring $\rho > 0$) in
$$|F(x)| \le \frac{K_1}{|x|^{d+2+\rho}}.$$
This extra assumption is satisfied for all known applications.
- The theorem directly applies to self-avoiding walk. For percolation, we combine the theorem with an elementary convolution estimate. We obtain
$$G_{z_c}(x), \tau_{p_c}(x) = \frac{\text{const}}{|x|^{d-2}} + O\left(\frac{1}{|x|^{d-\varepsilon}}\right)$$
with arbitrary $\varepsilon > 0$. Hara obtained $\varepsilon = 2 - 2/d$.

Gaussian deconvolution

- Our proof is completely different and is short and simple. It is inspired by the work of Slade on weakly self-avoiding walks in 2022. But to cover percolation, we need new ideas.
- The decay assumption on $F(x)$ can be replaced by regularity assumptions on $|x|^{2+\varepsilon}F(x)$ and $|x|^{d-2}F(x)$.
- (Ongoing) Extension to models on $\mathbb{R}^d$, e.g., random connection model.
- (Ongoing) Anisotropic $|x|^{-(d-2)}$ decay (using only $\mathbb{Z}_2$-symmetry).

Strategy of proof

Recall $G(x) = \int_{\mathbb{T}^d} \frac{e^{-ik\cdot x}}{\hat{F}(k)} \frac{dk}{(2\pi)^d}$ and $\kappa = -\sum_{x\in\mathbb{Z}^d} |x|^2 F(x) \in (0,\infty)$.

We decompose

$$\hat{G} := \frac{1}{\hat{F}} = \kappa^{-1}\frac{1}{1-\hat{D}} + \frac{(1-\hat{D}) - \kappa^{-1}\hat{F}}{(1-\hat{D})\hat{F}} = \kappa^{-1}\frac{1}{1-\hat{D}} + \frac{\hat{E}}{(1-\hat{D})\hat{F}},$$

where $E = A - \kappa^{-1}F$ with $A = \delta - D$. The constant κ is chosen to make

$$\sum_{x\in\mathbb{Z}^d} |x|^2 E(x) = 0,$$

so that the remainder would be more regular then the leading term.

By inverse Fourier transform, since $C(x) = \int_{\mathbb{T}^d} \frac{e^{-ik\cdot x}}{1-\hat{D}(k)} \frac{dk}{(2\pi)^d}$, we get

$$G(x) = \kappa^{-1}C(x) + f(x),$$

where f is the inverse Fourier transform of $\hat{f} := \hat{E}/(\hat{A}\hat{F})$.

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Strategy of proof

From

$$G(x) = \kappa^{-1}C(x) + f(x),$$

and

$$C(x) = \frac{a_d}{|x|^{d-2}} + O\left(\frac{1}{|x|^d}\right),$$

it suffices to prove $f(x) = o(|x|^{-(d-2)})$ as $|x| \to \infty$.

The choice of κ allows us to take $d-2$ (weak) derivatives of $\hat{f} = \hat{E}/(\hat{A}\hat{F})$. We will show all these derivatives are integrable, then by the Riemann–Lebesgue lemma, we get $|x|^{d-2}f(x) \to 0$ as $|x| \to \infty$.

Intuition

Since $E = A - \kappa^{-1} F$ is symmetric and satisfies

$$\sum_{x\in\mathbb{Z}^d} E(x) = \sum_{x\in\mathbb{Z}^d} |x|^2 E(x) = 0,$$

we roughly have $\nabla^\gamma \hat{E}(k) \lesssim |k|^{2+\sigma-|\gamma|}$ for some $\sigma \in (0, \min\{\rho, 2\})$.
By the assumed infrared bound, we have

$$\Big|\frac{1}{\hat{A}(k)}\Big|, \Big|\frac{1}{\hat{F}(k)}\Big| \lesssim \frac{1}{|k|^2}.$$

Taking derivatives roughly gives

$$\Big|\nabla^\gamma\Big(\frac{1}{\hat{A}(k)}\Big)\Big|, \Big|\nabla^\gamma\Big(\frac{1}{\hat{F}(k)}\Big)\Big| \lesssim \frac{1}{|k|^{2+|\gamma|}}.$$

Then by the product rule, we get

$$|\nabla^{d-2}\hat{f}| = \Big|\nabla^{d-2}\Big(\frac{\hat{E}}{\hat{A}\hat{F}}\Big)\Big| \lesssim \frac{|k|^{2+\sigma}}{|k|^{2+2+d-2}} = \frac{|k|^\sigma}{|k|^d} \in L^1(\mathbb{T}^d).$$

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- The intuition works for self-avoiding walk (Slade 2022) but does not work for percolation, where we cannot take enough classical derivatives ($\Pi(x)$ does not decay fast enough).
- Solution: We use weak derivatives, and replace power-counting by Hölder's inequality.

Weak derivative

Let $C_c^\infty(\mathbb{T}^d)$ denote the space of infinitely differentiable, compactly supported *test functions* $\phi : \mathbb{T}^d \to \mathbb{R}$. (For the torus $\mathbb{T}^d$, every function has compact support.)

Definition (Weak derivative)

Suppose $u, v \in L^1(\mathbb{T}^d)$ and α is a multi-index. We say that v is the α^{th} *weak partial derivative* of u, written $\nabla^\alpha u = v$, if, for all test functions $\phi \in C_c^\infty(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} u \nabla^\alpha \phi = (-1)^{|\alpha|} \int_{\mathbb{T}^d} v\phi.$$

The requirement is the usual integration by parts formula, so u is weakly differentiable if it is classically differentiable.

Lemma

The weak derivative satisfies the usual product and quotient rules, provided the result is integrable.

Weak derivative and Fourier transform

For us, we just need the fact that the L^2 Fourier transform gives the weak derivative. We write $\mathcal{F}[f] = \hat{f}$ for the L^2 Fourier transform of $f \in \ell^2(\mathbb{Z}^d)$.

Lemma

Let $f : \mathbb{Z}^d \to \mathbb{R}$ and α be a multi-index. Suppose $x^\alpha f(x) \in \ell^2(\mathbb{Z}^d)$. Then the α^{th} weak partial derivative of $\hat{f}$ is given by

$$\nabla^\alpha \hat{f} = \mathcal{F}[(ix)^\alpha f(x)].$$

We use the lemma to make sense of $\nabla^{d-2}\hat{F}(k)$. This is the origin of our restriction $\rho > \frac{d-8}{2}$ on

$$|F(x)| \leq \frac{K_1}{|x|^{d+2+\rho}};$$

we need $|x|^{d-2}F(x) \in \ell^2(\mathbb{Z}^d)$.

Proof of main result

We want to show $\hat{f} = \frac{\hat{E}}{\hat{A}\hat{F}}$ is $d-2$ times weakly differentiable. By the product and quotient rules, $\nabla^\alpha \hat{f}$ is given by a linear combination of terms of the form

$$\left(\prod_{n=1}^{i} \frac{\nabla^{\delta_n}\hat{A}}{\hat{A}}\right)\left(\frac{\nabla^{\alpha_2}\hat{E}}{\hat{A}\hat{F}}\right)\left(\prod_{m=1}^{j} \frac{\nabla^{\gamma_m}\hat{F}}{\hat{F}}\right),$$

where $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, $0 \leq i \leq |\alpha_1|$, $0 \leq j \leq |\alpha_3|$, $\sum_{n=1}^{i} \delta_n = \alpha_1$, and $\sum_{m=1}^{j} \gamma_m = \alpha_3$, *provided these terms are integrable.*

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where $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, $0 \le i \le |\alpha_1|$, $0 \le j \le |\alpha_3|$, $\sum_{n=1}^{i} \delta_n = \alpha_1$, and $\sum_{m=1}^{j} \gamma_m = \alpha_3$, *provided these terms are integrable.*

Lemma

Let $|\gamma| < \frac{1}{2}d + 2 + \rho$, *and choose* $\sigma \in (0, \rho)$ *such that* $\sigma \le 2$. *Then*

$$\frac{\nabla^\gamma \hat{A}}{\hat{A}}, \frac{\nabla^\gamma \hat{F}}{\hat{F}} \in L^q \quad (q^{-1} > \frac{|\gamma|}{d}), \qquad \frac{\nabla^\gamma \hat{E}}{\hat{A}\hat{F}} \in L^q \quad (q^{-1} > \frac{2-\sigma+|\gamma|}{d}).$$

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By Hölder's inequality,

$$\left(\prod_{n=1}^{i} \frac{\nabla^{\delta_n} \hat{A}}{\hat{A}} \right) \left(\frac{\nabla^{\alpha_2} \hat{E}}{\hat{A}\hat{F}} \right) \left(\prod_{m=1}^{j} \frac{\nabla^{\gamma_m} \hat{F}}{\hat{F}} \right) \in L^r(\mathbb{T}^d)$$

as long as

$$\frac{1}{r} > \frac{\sum_{n=1}^{i} |\delta_n|}{d} + \frac{2 - \sigma + |\alpha_2|}{d} + \frac{\sum_{m=1}^{j} |\gamma_m|}{d} = \frac{|\alpha| + 2 - \sigma}{d}.$$

Since $|\alpha| \leq d - 2$ and $\sigma > 0$, we can take $r = 1$. This proves that $\hat{f}$ is $d - 2$ times weakly differentiable and concludes the proof. □

Proof of lemma

Lemma

Let $|\gamma| < \frac{1}{2}d + 2 + \rho$, *and choose* $\sigma \in (0, \rho)$ *such that* $\sigma \leq 2$. *Then*

$$\frac{\nabla^\gamma \hat{A}}{\hat{A}}, \frac{\nabla^\gamma \hat{F}}{\hat{F}} \in L^q \quad (q^{-1} > \frac{|\gamma|}{d}), \qquad \frac{\nabla^\gamma \hat{E}}{\hat{A}\hat{F}} \in L^q \quad (q^{-1} > \frac{2 - \sigma + |\gamma|}{d}).$$

Bound on $\nabla^\gamma \hat{A}/\hat{A}$.

Recall $A = \delta - D$ has finite support. If $|\gamma| = 1$, by Taylor's theorem and symmetry, we have $|\nabla^\gamma \hat{A}(k)| \lesssim |k|$. If $|\gamma| \geq 2$, Taylor's theorem gives $|\nabla^\gamma \hat{A}(k)| \lesssim 1$ instead. Together with the infrared bound, we get

$$\Big|\frac{\nabla^\gamma \hat{A}}{\hat{A}}(k)\Big| \lesssim \frac{1}{|k|^{\min(|\gamma|,2)}} \in L^q(\mathbb{T}^d) \qquad (q^{-1} > \frac{\min(|\gamma|, 2)}{d}),$$

which is stronger than the desired result. □

Proof of lemma

Bound on $\nabla^\gamma \hat{F}/\hat{F}$.

The $|\gamma| = 1$ case is the same as for $\hat{A}$, because $\sum_x |x|^2 |F(x)|$ is finite. For $|\gamma| \geq 2$, the decay assumption $|F(x)| \lesssim |x|^{-(d+2+\rho)}$ and boundedness of the Fourier transform imply

$$\nabla^\gamma \hat{F} \in L^{\frac{d}{|\gamma|-2}}(\mathbb{T}^d) \qquad (2 \leq |\gamma| < \tfrac{1}{2}d + 2 + \rho).$$

Since $|\hat{F}^{-1}(k)| \lesssim |k|^{-2} \in L^p$ for all $p^{-1} > 2/d$ by the infrared bound, it follows from Hölder's inequality that $\nabla^\gamma \hat{F}/\hat{F} \in L^q$ for all $q^{-1} > (|\gamma| - 2 + 2)/d$, as desired. □

Proof of lemma

Bound on $\nabla^\gamma \hat{E}/(\hat{A}\hat{F})$.

Let $\sigma \in (0, \rho)$ be such that $\sigma \le 2$. We use the fact that $E = A - \kappa^{-1} F$ has the same $|x|^{-(d+2+\rho)}$ decay as F. If $|\gamma| < 2 + \sigma$, it follows from

$$\sum_{x \in \mathbb{Z}^d} E(x) = \sum_{x \in \mathbb{Z}^d} |x|^2 E(x) = 0,$$

symmetry, and infrared bounds that

$$\Big| \frac{\nabla^\gamma \hat{E}}{\hat{A}\hat{F}}(k) \Big| \lesssim \frac{|k|^{2+\sigma-|\gamma|}}{|k|^2 |k|^2} = \frac{1}{|k|^{2-\sigma+|\gamma|}},$$

which is in L^q for $q^{-1} > (2 - \sigma + |\gamma|)/d$, as desired.

If $|\gamma| \ge 2 + \sigma$, we use the Fourier transform to bound $\nabla^\gamma \hat{E}$, then use Hölder's inequality (as in the $|\gamma| \ge 2$ case for $\hat{F}$). □

This concludes the proof of the lemma.

Main result (revisit)

Theorem (Hara'08, L.–Slade'23)

Let $d > 2$. *Suppose* $F : \mathbb{Z}^d \to \mathbb{R}$ *is a* $\mathbb{Z}^d$*-symmetric function, and suppose there are* $K_1, K_2 > 0$, $\rho > \max(0, \frac{d-8}{2})$ *such that, for all* $x \in \mathbb{Z}^d$ *and* $k \in \mathbb{T}^d$,

$$|F(x)| \le \frac{K_1}{|x|^{d+2+\rho}}, \qquad \hat{F}(0) = 0, \qquad \hat{F}(k) - \hat{F}(0) \ge K_2 |k|^2.$$

Then

$$G(x) \sim \frac{a_d}{\kappa |x|^{d-2}} \qquad \text{as } |x| \to \infty,$$

where $\kappa = -\sum_{x \in \mathbb{Z}^d} |x|^2 F(x) \in (0, \infty)$.

We have proved $G(x) = \kappa^{-1} C(x) + f(x)$ and $\nabla^{d-2} \hat{f} \in L^1(\mathbb{T}^d)$.
Error estimate?

Better error estimate

We can improve the error to $f(x) = O(|x|^{-(d-2+\delta)})$, $\delta > 0$, by taking more derivatives of $\hat{f}$.
For fractional powers of $|x|$, we use the following integral representation: For $\delta \in (0,1)$,

$$(\operatorname{sgn} x_1)|x_1|^\delta = \frac{1}{c_\delta}\int_0^\infty \frac{\sin(x_1 u)}{u^{1+\delta}}du, \quad c_\delta = \int_0^\infty \frac{\sin u}{u^{1+\delta}}du \in (0,\infty).$$

Multiplying by $\sin(x_1 u)$ produces phase shifts in the Fourier space.

Lemma (Fractional derivative)

Let $\tilde{u} = (u, 0, \ldots, 0)$. *Suppose that* $\hat{g} \in L^1(\mathbb{T}^d)$ *and that*

$$\frac{1}{2ic_\delta}\int_0^\infty \frac{1}{u^{1+\delta}}\left\|\hat{g}(\cdot + \tilde{u}) - \hat{g}(\cdot - \tilde{u})\right\|_{L^1(\mathbb{T}^d)}du < \infty.$$

Then $\sup_{x\in\mathbb{Z}^d}|x_1|^\delta|g(x)| < \infty$.

Better error estimate

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Then $\sup_{x \in \mathbb{Z}^d} |x_1|^\delta |g(x)| < \infty$.

We use the lemma with $\hat{g} = \nabla^\alpha \hat{f}$ where $|\alpha| = d - 2$.
Write

$$(U_u \hat{g})(k) = \hat{g}(k + \tilde{u}) - \hat{g}(k - \tilde{u}).$$

Estimates on $U_u(\nabla^\alpha \hat{f})$ then lead to more decay of $f(x)$.

Estimates on $U_u(\nabla^\alpha \hat{f})$

Since $\nabla^\alpha \hat{f}$ is given by a linear combination of terms of the form

$$\left(\prod_{n=1}^{i} \frac{\nabla^{\delta_n}\hat{A}}{\hat{A}}\right)\left(\frac{\nabla^{\alpha_2}\hat{E}}{\hat{A}\hat{F}}\right)\left(\prod_{m=1}^{j} \frac{\nabla^{\gamma_m}\hat{F}}{\hat{F}}\right),$$

and U_u is taking a finite difference, we apply U_u to one factor at a time.

Lemma

Let γ be a multi-index, $0 \le \eta \le 1$, with $|\gamma| + \eta < \frac{1}{2}d + 2 + \rho$. Choose $\sigma \in (0, \rho)$ such that $\sigma \le 2$, and choose q_1, q_2 satisfying

$$q_1^{-1} > \frac{|\gamma| + \eta}{d}, \qquad q_2^{-1} > \frac{2 - \sigma + |\gamma| + \eta}{d}.$$

Then for $0 \le u \le 1$,

$$\left\|U_u\left(\frac{\nabla^\gamma \hat{A}}{\hat{A}}\right)\right\|_{q_1}, \ \left\|U_u\left(\frac{\nabla^\gamma \hat{F}}{\hat{F}}\right)\right\|_{q_1}, \ \left\|U_u\left(\frac{\nabla^\gamma \hat{E}}{\hat{A}\hat{F}}\right)\right\|_{q_2} \lesssim u^\eta.$$

Estimates on $U_u(\nabla^\alpha \hat{f})$: an ingredient

$$(U_u\hat{g})(k) = \hat{g}(k+\tilde{u}) - \hat{g}(k-\tilde{u}).$$

Lemma ("Sobolev inequality")

Let $g : \mathbb{T}^d \to \mathbb{C}$ be weakly differentiable. Fix $1 \le p < d$. Assume $\nabla^{e_j} g \in L^p(\mathbb{T}^d)$ for all j. Let $0 \le \eta \le 1$ and define p_η by $\frac{1}{p_\eta} = \frac{1}{p} - \frac{1-\eta}{d}$. Then

$$\|U_u g\|_{p_\eta} \lesssim u^\eta \|g\|_{W^{1,p}},$$

where $\|g\|_{W^{1,p}} = (\|g\|_p^p + \sum_{j=1}^d \|\nabla_j g\|_p^p)^{1/p}$.

References

- Y. Liu and G. Slade. Gaussian deconvolution and the lace expansion. Preprint, arXiv:2310.07635.
- T. Hara. Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. *Ann. Probab.*, **36**:530–593, (2008).
- G. Slade. A simple convergence proof for the lace expansion. *Ann. I. Henri Poincaré Probab. Statist.*, **58**:26–33, (2022).

If you want to learn lace expansion for spread-out models:

- Y. Liu and G. Slade. Gaussian deconvolution and the lace expansion for spread-out models. Preprint, arXiv:2310.07640.

Thank You!

Near-critical upper bound

The method can be extended to study a family of convolution equations,

$$F_z * G_z = \delta.$$

With F_z satisfying similar "massive" assumptions, we prove the uniform upper bound

$$G_z(x) \le \frac{c_0}{\max(1, |x|^{d-2})} e^{-c_1 m(z)|x|},$$

where $m(z)$ is the exponential decay rate of $G_z(x)$, for $z \in [z_c - \delta, z_c)$, $\delta > 0$.
The result applies to strictly self-avoiding walk in dimensions $d > 4$.

Reference: Y. Liu. A general approach to massive upper bound for two-point function with application to self-avoiding walk torus plateau. Preprint, arXiv:2310.17321.