Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2026, Vol. 62, No. 1, 68–85
https://doi.org/10.1214/24-AIHP1509


ANNALES
DE L'INSTITUT
HENRI
POINCARÉ
PROBABILITÉS
ET STATISTIQUES

# Gaussian deconvolution and the lace expansion for spread-out models

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Received 24 October 2023; revised 16 May 2024; accepted 16 May 2024

**Abstract.** We present a new proof of $|x|^{-(d-2)}$ decay of critical two-point functions for spread-out statistical mechanical models on $\mathbb{Z}^d$ above the upper critical dimension, based on the lace expansion and assuming appropriate diagrammatic estimates. Applications include spread-out models of the Ising model and self-avoiding walk in dimensions $d > 4$, and spread-out percolation for $d > 6$. The proof is based on an extension of the new Gaussian deconvolution theorem we obtained in a recent paper. It provides a technically simpler and conceptually more transparent approach than the method of Hara, van der Hofstad and Slade (*Ann. Probab.* **31** (2003) 349–408).

**Résumé.** Nous présentons une nouvelle preuve de la décroissance en $|x|^{-(d-2)}$ des fonctions à deux points critiques pour les modèles mécaniques statistiques étendus sur $\mathbb{Z}^d$ au-dessus de la dimension critique supérieure, basée sur le développement en lacets et en supposant des estimations diagrammatiques appropriées. Les applications incluent des modèles étendus du modèle d'Ising et de la marche auto-évitante en dimensions $d > 4$, ainsi que la percolation étendue pour $d > 6$. La preuve est basée sur une extension du nouveau théorème de déconvolution gaussienne que nous avons obtenu dans un article récent. Elle fournit une approche techniquement plus simple et conceptuellement plus transparente que la méthode de Hara, van der Hofstad et Slade (*Ann. Probab.* **31** (2003) 349–408).

*MSC2020 subject classifications:* 42B05; 60K35; 82B27; 82B41; 82B43

*Keywords:* Convolution; Fourier transform; Weak derivative; Random walk; Lace expansion; Spin system; Self-avoiding walk; Percolation; Lattice trees; Lattice animals

## 1. Introduction and results

### 1.1. *Introduction*

The lace expansion has been used to prove mean-field behaviour for several statistical mechanical models above their upper critical dimension $d_c$, including self-avoiding walk ($d_c = 4$), the Ising model ($d_c = 4$), percolation ($d_c = 6$), and lattice trees and lattice animals ($d_c = 8$). The expansion requires a small parameter for its convergence, which roughly speaking is $(d - d_c)^{-1}$ for nearest-neighbour models, so $d$ is required to be large, e.g., $d \geq 11$ for percolation [3]. (An exception is self-avoiding walk for which dimensions $d \geq 5$ have been handled with computer assistance [7].) This obscures the role of the upper critical dimension, and in order to apply lace expansion methods to analyse critical behaviour in all dimensions $d > d_c$, spread-out models were first used in [6]. Spread-out models extend nearest-neighbour connections to include long connections (finite-range or rapidly decaying) parametrised by a large parameter $L \gg 1$. The reciprocal of $L$ provides a small parameter that can be used to obtain convergence of the lace expansion without the need to take the dimension artificially large. In addition, the proof of mean-field behaviour for a wide class of spread-out models serves as a demonstration of universality.

In [5,15], $|x|^{-(d-2)}$ decay of critical two-point functions was proved for spread-out versions of all the above-mentioned models above $d_c$ (i.e., the critical exponent $\eta$ is equal to zero), and this decay has important applications, e.g., [9,11,12] for percolation. The proofs in [5,15] are based on a bootstrap argument and a deconvolution theorem developed in [5]. The method of [5] involves intricate Fourier analysis, and our purpose in this paper is to provide a technically simpler and conceptually clearer replacement for the method of [5]. Our method is based on the deconvolution theorem of [14], which itself was inspired by [17]. The deconvolution theorem of [14] provides a new and simpler proof of Hara's Gaussian Lemma [4], using only elementary facts about the Fourier transform, product and quotient rules of differentiation, and Hölder's inequality. We adapt it here to apply to spread-out models. In the process, we rely on results proved in [14], particularly those concerning the $L^p$ theory of the Fourier transform in the context of weak derivatives.

The adaptation requires control of the $L$-dependence in estimates, which feeds into the bootstrap argument that is central to the convergence proof for the lace expansion. The general deconvolution theorem of [14] does not, on its own, provide sufficient control in error bounds to effectively deal with this interrelation. For this reason, we also develop and present a generic approach to the bootstrap analysis in the context of spread-out models.

For self-avoiding walk, the lace expansion [2,16] produces a convolution equation for the two-point function $G$, of the form

$$F * G = \delta \tag{1.1}$$

with $(F * G)(x) = \sum_{y\in\mathbb{Z}^d} F(x-y)G(y)$, $\delta$ the Kronecker delta $\delta(x) = \delta_{0,x}$, and $F$ explicitly defined by the lace expansion. We refer to (1.1) as the *impulse* equation. Our deconvolution theorem gives conditions on $F$ which guarantee $G(x)$ to have $|x|^{-(d-2)}$ decay at criticality. For percolation or the Ising model, the lace expansion [6,8,15] instead produces an inhomogeneous convolution equation of the form

$$\tilde{F} * G = h, \tag{1.2}$$

with explicit functions $\tilde{F}$ and $h$. We will prove that in the spread-out setting, (1.2) can be reduced to the simpler (1.1), so that our deconvolution theorem then extends to (1.2). In [5], (1.1) is instead rewritten in the form (1.2) in order to handle both equations simultaneously. Our reduction is more efficient, as it reduces the more difficult equation to the simpler one, rather than vice versa.

*Notation.* We write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. We write $f = O(g)$ or $f \lesssim g$ to mean there exists a constant $C > 0$ such that $|f(x)| \le C|g(x)|$, and $f = o(g)$ for $\lim f/g = 0$. To avoid dividing by zero, with $|x|$ the Euclidean norm of $x \in \mathbb{R}^d$, we define

$$|\!|\!| x |\!|\!| = \max\{|x|, 1\}. \tag{1.3}$$

Note that (1.3) does not define a norm on $\mathbb{R}^d$.

*Fourier transform.* Let $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ denote the continuum torus, which we identify with $(-\pi, \pi]^d \subset \mathbb{R}^d$. For a function $f \in L^1(\mathbb{Z}^d)$, the Fourier transform and its inverse are given by

$$\hat{f}(k) = \sum_{x\in\mathbb{Z}^d} f(x)e^{ik\cdot x} \quad \big(k \in \mathbb{T}^d\big), \qquad f(x) = \int_{\mathbb{T}^d} \hat{f}(k)e^{-ik\cdot x}\frac{dk}{(2\pi)^d} \quad \big(x \in \mathbb{Z}^d\big). \tag{1.4}$$

## 1.2. *Main results*

### 1.2.1. *Spread-out random walk*

Our point of reference is the Green function (two-point function) of a spread-out random walk on $\mathbb{Z}^d$, whose transition probability $D : \mathbb{Z}^d \to [0, 1]$ is given by the following definition.

**Definition 1.1.** Let $v : \mathbb{R}^d \to [0, \infty)$ be bounded, $\mathbb{Z}^d$-symmetric (invariant under reflection in coordinate hyperplanes and rotation by $\pi/2$), supported in $[-1, 1]^d$, with $\frac{\partial^d v}{\partial x_1 \cdots \partial x_d}$ piecewise continuous, and with $\int_{[-1,1]^d} v(x)\, dx = 1$. Given $v$, we define $D : \mathbb{Z}^d \to [0, 1]$ by

$$D(x) = \frac{v(x/L)}{\sum_{x\in\mathbb{Z}^d} v(x/L)}. \tag{1.5}$$

By definition, $D$ is always supported in $[-L, L]^d$. For example, if $v(x) = 2^{-d}\mathbb{1}\{\|x\|_\infty \le 1\}$, then $D$ is the uniform distribution on $[-L, L]^d \cap \mathbb{Z}^d$.

*Large L assumption.* For the above definition to make sense, the denominator in (1.5) must be nonzero. It will be nonzero if $L$ is large enough (depending on $d$ and $v$), since $\sum_{x\in\mathbb{Z}^d} v(x/L) \sim L^d$ as $L\to\infty$ by a Riemann sum approximation to $\int_{[-1,1]^d} v(x)\,dx$. Similarly, the variance $\sigma^2 = \sum_{x\in\mathbb{Z}^d} |x|^2 D(x)$ of $D$ as is asymptotic to a multiple of $L^2$. We assume throughout the entire paper that $L \geq L_0$ with $L_0$ chosen large enough that the denominator of (1.5) is nonzero. Moreover, when required we will increase the value of $L_0$, but only in a manner that depends on $d$ and $v$ alone. Since we work throughout with a fixed dimension $d$ and a fixed function $v$, we do not track the dependence on $d$ or $v$ of constants in bounds: all constants in bounds are permitted to depend on $d$ and $v$.

For $\mu\in[0,1]$, we let $S_\mu$ denote the *spread-out Green function*, which is the solution of the convolution equation

$$(\delta - \mu D) * S_\mu = \delta \tag{1.6}$$

given by

$$S_\mu(x) = \int_{\mathbb{T}^d} \frac{e^{-ik\cdot x}}{1-\mu\hat{D}(k)} \frac{dk}{(2\pi)^d} \quad (d>2). \tag{1.7}$$

When $D(x)$ is replaced by $P(x) = \frac{1}{2d}\mathbb{1}_{|x|=1}$, we have the nearest-neighbour random walk, and we denote its Green function by $C_\mu(x)$. When $\mu\in(0,1)$, both $S_\mu$ and $C_\mu$ decay exponentially. In the critical case $\mu=1$, it is well-known (e.g., [13,18]) that

$$C_1(x) = \frac{a_d}{|||x|||^{d-2}} + O\left(\frac{1}{|||x|||^d}\right), \qquad a_d = \frac{d\Gamma(\frac{d-2}{2})}{2\pi^{d/2}} \quad (d>2). \tag{1.8}$$

For $S_1(x)$, we establish a similar statement in Proposition 1.2.

**Proposition 1.2 (Spread-out Green function).** *Let $d>2$, $\varepsilon>0$, and $L\geq L_0$. Then*

$$S_1(x) = \delta_{0,x} + \frac{1}{\sigma^2}C_1(x) + O\left(\frac{1}{L^{1-\varepsilon}|||x|||^{d-1}}\right), \tag{1.9}$$

*with the constant uniform in L but dependent on $\varepsilon$. Also, there is a constant $K_S = K_S(\varepsilon)$ such that*

$$S_1(x) \leq \delta_{0,x} + K_S L^{-(2-\varepsilon)} |||x|||^{-(d-2)} \quad \left(x\in\mathbb{Z}^d\right). \tag{1.10}$$

Proposition 1.2 shows that $S_1(x)$ has the same $|x|^{-(d-2)}$ decay as $C_1(x)$. When the constant in the error term is permitted to depend on $L$, Proposition 1.2 is a standard result and is known to hold with error $O(|||x|||^{-d})$ like (1.8), e.g., [13,18]. However, the uniformity in $L$ needs care, is not standard, and is required for our results. It is proved in [5, Proposition 1.6], using intricate Fourier analysis, that

$$S_1(x) = \frac{a_d}{\sigma^2}\frac{1}{|||x|||^{d-2}} + O\left(\frac{1}{|||x|||^{d-2+s}}\right), \tag{1.11}$$

for any $s<2$, with the constant independent of $L$. For the case $s=1$, our error estimate in (1.9) has a good power of $L$ compared to (1.11). We prove Proposition 1.2 using the simple strategy introduced in [14], as an alternative to the proof in [5]. Via the fractional derivative analysis of [14, Section 2.3], the error term in (1.9) can in fact be improved to $O(L^{-(2-t)}|||x|||^{-(d-2+s)})$ with any $0\leq s<t<2$ and with the constant independent of $L$. We omit the details of this improvement, which we do not need or invoke later.

1.2.2. *Spread-out Gaussian deconvolution theorem*

We now turn to general spread-out models. The following assumption isolates basic properties that are typical of the two-point function of models such as self-avoiding walk or percolation. Our restriction to $z_c \geq 1$ is a convenience that can be achieved by scaling the model's parameter (e.g., the bond occupation probability $p$ for percolation).

**Assumption 1.3.** Let $d>2$. We assume that $G_z : \mathbb{Z}^d \to [0,\infty]$, defined for $z\in[1,z_c]$ with some $z_c\geq 1$, is a family of $\mathbb{Z}^d$-symmetric functions such that:

(i) $\sum_{x\in\mathbb{Z}^d} G_{z_c}(x) = \infty$,
(ii) for each $z<z_c$, $\sum_{x\in\mathbb{Z}^d} G_z(x) < \infty$ and $G_z(x) = o(|x|^{-(d-2)})$ as $|x|\to\infty$ (need not be uniform in $z$),

(iii) for each $x$, $G_z(x)$ is non-decreasing and continuous in $z \in [1, z_c]$,
(iv) $G_1(x) \le S_1(x)$ for all $x \in \mathbb{Z}^d$.

It is not part of the assumption that the critical $G_{z_c}(x)$ is finite, and the goal is to prove that $G_{z_c}(x)$ has Gaussian $|x|^{-(d-2)}$ decay. To do this, we first establish a uniform in $z < z_c$ bound using the model-independent bootstrap argument of [5], and then we bound $G_{z_c}$ using monotone convergence. The bootstrap argument compares $G_z$ to an upper bound of $S_1$, as follows. Let $d > 2$, fix a small $\varepsilon > 0$, and let $K_S = K_S(\varepsilon)$ be the constant in the bound (1.10). Given $L$, we define the *bootstrap function* $b : [1, z_c] \to [0, \infty]$ by

$$b(z) = \max\left\{\sup_{x \neq 0} \frac{G_z(x)}{K_S L^{-2+\varepsilon}|x|^{-(d-2)}}, 3(z-1)\right\}. \tag{1.12}$$

The function $b(z)$ is finite and continuous in $z \in [1, z_c)$ by Assumption 1.3(ii)–(iii), and $b(1) \le 1$ by Assumption 1.3(iv).

The next assumption gives consequences of an *a priori* bound $b(z) \le 3$. It reflects the fact that when $b(z) \le 3$, diagrams arising from the lace expansion, which are functionals of $G_z$, can be bounded by functionals of the explicit function $L^{-2+\varepsilon}|x|^{-(d-2)}$. The verification of Assumption 1.4 is model dependent and is carried out for spread-out models of self-avoiding walk, Ising model, percolation, and lattice trees and lattice animals, above their upper critical dimensions, in [5,15]; its verification requires large $L$. We do not verify Assumption 1.4 here. We will later prove that $b(z) \le 2$ for all $z \le z_c$ via the bootstrap argument, again assuming large $L$. The specific $L$-dependence in the upper bound of (1.14) plays an important role in the bootstrap, and the proof of $b(z) \le 2$ requires that dependence to mesh well with the $L$-dependence in the upper bound (1.10) on $S_1(x)$. This is a subtlety that does not occur in [14] but that must be dealt with for spread-out models.

**Assumption 1.4 (Lace expansion).** Let $d \ge 1$, let $D$ be given by Definition 1.1, and let $\rho > \frac{d-8}{2} \vee 0$. If $b(z) \le 3$ then there exists a $\mathbb{Z}^d$-symmetric function $\Pi_z : \mathbb{Z}^d \to \mathbb{R}$ for which

$$G_z = \delta + zD * G_z + \Pi_z * G_z, \tag{1.13}$$

and

$$\left|\Pi_z(x)\right| \le K_\Pi L^{-2+\epsilon}\left(\delta_{0,x} + \frac{o(1)}{\|\!|x|\!\|^{d+2+\rho}}\right) \quad \left(x \in \mathbb{Z}^d\right), \tag{1.14}$$

where the constant $K_\Pi$ is independent of $z$ and $L$, and $o(1) \to 0$ uniformly in $z$ and $x$ as $L \to \infty$.

Equation (1.13) is what the lace expansion produces for self-avoiding walk, and it can be rewritten as the impulse equation

$$(\delta - zD - \Pi_z) * G_z = \delta. \tag{1.15}$$

For percolation and the Ising model, the lace expansion instead produces an inhomogeneous convolution equation

$$G_z = h_z + zD * h_z * G_z \quad \iff \quad (\delta - zD * h_z) * G_z = h_z, \tag{1.16}$$

with an explicit function $h_z$. The next assumption covers these models with inhomogeneous convolution equation.

**Assumption 1.5 (Inhomogeneous lace expansion).** Let $d \ge 1$, let $D$ be given by Definition 1.1 and let $\rho > \frac{d-8}{2} \vee 0$. If $b(z) \le 3$ then there exists a $\mathbb{Z}^d$-symmetric function $h_z : \mathbb{Z}^d \to \mathbb{R}$ for which

$$G_z = h_z + zD * h_z * G_z, \tag{1.17}$$

and

$$\left|h_z(x) - \delta_{0,x}\right| \le K_h L^{-2+\epsilon}\left(\delta_{0,x} + \frac{o(1)}{\|\!|x|\!\|^{d+2+\rho}}\right) \quad \left(x \in \mathbb{Z}^d\right), \tag{1.18}$$

where the constant $K_h$ is independent of $z$ and $L$, and $o(1) \to 0$ uniformly in $z$ and $x$ as $L \to \infty$.

**Proposition 1.6.** *Let $d \geq 1$. Suppose $h_z : \mathbb{Z}^d \to \mathbb{R}$ is $\mathbb{Z}^d$-symmetric and satisfies (1.18) and its uniformity assumptions with $\rho > -2$. Then there is an $L_1$ (depending on $K_h$, $\varepsilon$, and the $o(1)$) such that, for all $L \geq L_1$, there exists a $\mathbb{Z}^d$-symmetric function $\Phi_z : \mathbb{Z}^d \to \mathbb{R}$ for which $G_z$ obeying (1.17) satisfies $F_z * G_z = \delta$ with*

$$F_z = \delta - zD - \Phi_z, \qquad \left|\Phi_z(x)\right| \leq K_\Phi L^{-2+\varepsilon}\left(\delta_{0,x} + \frac{o(1)}{\| x \|^{d+2+\rho}}\right) \quad \left(x \in \mathbb{Z}^d\right), \tag{1.19}$$

*with the constant $K_\Phi$ independent of $z$ and $L$, and $o(1) \to 0$ uniformly in $z$ and $x$ as $L \to \infty$.*

In brief, Proposition 1.6 shows that Assumption 1.5 implies Assumption 1.4, with $\Phi_z$ playing the role of $\Pi_z$: the inhomogeneous equation (1.16) can be rewritten in the form of the impulse equation (1.15) when $h_z$ obeys (1.18). The following theorem is our main result. A minor additional assumption allows its hypothesis $d > 4$ to be weakened to $d > 2$; see Remark 2.4.

**Theorem 1.7.** *Let $d > 4$. Under Assumption 1.3, together with either Assumption 1.4 or Assumption 1.5, there is an $L_2$ such that for all $L \geq L_2$,*

$$G_{z_c}(x) \sim \frac{\lambda_{z_c}}{\sigma^2}\frac{a_d}{|x|^{d-2}} \quad \text{as } |x| \to \infty, \tag{1.20}$$

*where $a_d$ is the constant of (1.8), and, for any fixed $\varepsilon > 0$, $\lambda_{z_c} = 1 + O(L^{-2+\varepsilon})$.*

The constant $\lambda_{z_c}$ is given explicitly in (2.17) in terms of $\Pi_{z_c}$. Theorem 1.7 is essentially proved in [5], with an explicit error term. Our contribution here is to provide a different proof based on the simple strategy of [14], which replaces the more difficult and less conceptual analysis in [5, p.358, items (i) and (iv)].

It is part of the proof of Theorem 1.7 that the bootstrap function obeys $b(z_c) \leq 2$, so that Assumption 1.4 immediately verifies the hypotheses of [14, Theorem 1.2]. That theorem then immediately improves (1.20) with an explicit error term:

$$G_{z_c}(x) = \frac{\lambda_{z_c}}{\sigma^2}\frac{a_d}{\| x \|^{d-2}} + \frac{O_L(1)}{\| x \|^{d-2+s}}, \tag{1.21}$$

with any $s < \rho \wedge 2 \wedge (\rho - \frac{d-8}{2})$. In applications, there is generally a sufficiently large but fixed value of $L$, and $L$-dependence of the error term in (1.21) is of no importance. On the other hand, [14, Theorem 1.2] cannot be applied *ab initio*, because some control of $L$-dependence of the error term is needed in order to complete the bootstrap argument used to prove Theorem 1.7. This delicate point will materialise below, in Theorem 2.2 and in the proof of Proposition 2.3.

The verification of Assumption 1.4 or 1.5 is model-specific and requires $d > d_c$, where

$$d_c = \begin{cases} 4 & \text{(self-avoiding walk, Ising)}, \\ 6 & \text{(percolation)}, \\ 8 & \text{(lattice trees/animals)}. \end{cases} \tag{1.22}$$

For each of these models, Assumption 1.4 or 1.5 has been verified for large $L$ in [5,15] with

$$\rho = \begin{cases} 2(d-4) & \text{(self-avoiding walk, Ising)}, \\ d-6 & \text{(percolation)}, \\ d-8 & \text{(lattice trees/animals)}. \end{cases} \tag{1.23}$$

The requirement that $\rho > \frac{d-8}{2} \vee 0$ is satisfied in all cases.

### 1.3. *Organisation*

In Section 2, we prove our main result Theorem 1.7 subject to a general Gaussian deconvolution theorem (Theorem 2.2), Proposition 1.2, and Proposition 1.6. The proof of Theorem 2.2 is given in Section 3, and the proof of Proposition 1.2 is given in Appendix A. Both proofs use the simple strategy of [14]. In Section 4, we introduce a novel way to reduce the inhomogeneous lace expansion convolution equation (1.16) for percolation, Ising model and lattice trees/animals, to the self-avoiding walk equation (1.15), thereby proving Proposition 1.6. Finally, Appendix B provides a proof of some useful properties of the transition probability $D$.

## 2. Proof of Theorem 1.7

In Section 2.1, we state a general Gaussian deconvolution theorem (Theorem 2.2) in the spread-out setting. The theorem involves an interval of $z$ values $[1,\infty)$ and functions $F_z : \mathbb{Z}^d \to \mathbb{R}$, and its statement concerns the large-$x$ behaviour of the Fourier integral

$$\tilde{G}_z(x) = \int_{\mathbb{T}^d} \frac{e^{-ik\cdot x}}{\hat{F}_z(k)} \frac{dk}{(2\pi)^d}. \tag{2.1}$$

Our assumption on $F_z$ is motivated by (1.14) and (1.15). In Section 2.2, we prove our main result Theorem 1.7 using Theorem 2.2 and a generic bootstrap argument. Part of the proof involves verifying that under Assumptions 1.3 and 1.4, $G_z$ is indeed equal to the Fourier integral $\tilde{G}_z$.

### 2.1. *Gaussian deconvolution*

The assumption on $F_z$ is the following. It involves generic parameters $\beta_0$, $\beta_1$ in place of the specific $L$-dependent choices made in (1.14). The *a priori* bound $b(z) \le 3$ in Assumption 1.4 is absent here, but in Section 2.2 it will reappear.

**Assumption 2.1.** Suppose that $D$ is given by Definition 1.1 and that $z \ge 1$. We assume $F_z$ is given by $F_z = \delta - zD - \Pi_z$, where $\Pi_z$ is a $\mathbb{Z}^d$-symmetric function that satisfies

$$\left|\Pi_z(x)\right| \le \beta_0 \delta_{0,x} + \frac{\beta_1}{|\!|\!| x |\!|\!|^{d+2+\rho}} \tag{2.2}$$

with some $\rho > \frac{d-8}{2} \vee 0$ and $\beta = \beta_0 \vee \beta_1 \ge 0$ sufficiently small. Suppose also that $\hat{F}_z(0) \ge 0$.

As we will see in (3.23), Assumption 2.1 implies an *infrared bound* for $F_z$, namely that there is a constant $K_{\mathrm{IR}} > 0$ for which

$$\hat{F}_z(k) - \hat{F}_z(0) \ge K_{\mathrm{IR}}\big(L^2|k|^2 \wedge 1\big) \quad \big(k \in \mathbb{T}^d\big). \tag{2.3}$$

In dimensions $d > 2$, the infrared bound implies absolute convergence of the Fourier integral (2.1).

Define

$$\lambda_z = \frac{1}{\hat{F}_z(0) - \sigma^{-2}\sum_{x\in\mathbb{Z}^d}|x|^2 F_z(x)}, \qquad \mu_z = 1 - \lambda_z \hat{F}_z(0). \tag{2.4}$$

By Assumption 2.1, we have $\hat{F}_z(0) = 1 - z - \hat{\Pi}_z(0) \ge 0$. Since also $z \ge 1$, this implies that

$$1 \le z \le 1 - \hat{\Pi}_z(0) \le 1 + O(\beta). \tag{2.5}$$

It follows that $0 \le \hat{F}_z(0) \le O(\beta)$, and hence

$$\lambda_z = \frac{1}{\hat{F}_z(0) + z + \sigma^{-2}\sum_{x\in\mathbb{Z}^d}|x|^2\Pi_z(x)} = \frac{1}{1 + O(\beta) + O(\beta\sigma^{-2})} = 1 + O(\beta). \tag{2.6}$$

Since $0 \le \lambda_z \hat{F}_z(0) \le O(\beta)$, we have $\mu_z \in [1 - O(\beta), 1]$, so it makes sense to write $S_{\mu_z}$ (recall (1.6)). We also have $\mu_z \ge 1/2$ because $\beta$ is small. Given $\rho > \frac{d-8}{2} \vee 0$, we define

$$n_d = \begin{cases} d-2 & \left(\rho \le 1 + \left(\frac{d-8}{2} \vee 0\right)\right), \\ d-1 & \left(\rho > 1 + \left(\frac{d-8}{2} \vee 0\right)\right). \end{cases} \tag{2.7}$$

**Theorem 2.2 (Gaussian deconvolution).** *Let $d > 2$ and let $F_z$ satisfy Assumption 2.1. Then there exists a constant $c > 0$ such that the Fourier integral $\tilde{G}_z$ of (2.1) satisfies*

$$\tilde{G}_z(x) = \lambda_z S_{\mu_z}(x) + \begin{cases} O(\beta) & (x = 0), \\ O\left(\frac{\beta(L^{-c} + \beta) + \beta_1}{|x|^{n_d}}\right) & (x \ne 0), \end{cases} \tag{2.8}$$

*with the constants in the error term independent of $z$, $\beta_0$, $\beta_1$, $L$. Moreover, for fixed $z$, $\beta_0$, $\beta_1$, $L$, the error term in (2.8) can be replaced by $o(|x|^{-n_d})$ as $|x| \to \infty$.*

Theorem 2.2 is related to [5, Theorem 1.2]. The critical case of Theorem 2.2 can be inferred from the proof of [5, (1.13)], with $\beta$ taken to be of order $L^{-2+\varepsilon}$, $\varepsilon \in (0, \frac{\rho\wedge 1}{2})$ (as in the proof of [5, Proposition 2.2]), and with the error term in (2.8) replaced by

$$\frac{1}{L^{2-(\rho\wedge 2)} |||x|||^{d-2+s}} = \frac{\beta L^{(\rho\wedge 2)-\varepsilon}}{|||x|||^{d-2+s}} \quad (0 \le s < \rho \wedge 2). \tag{2.9}$$

For $s = n_d - (d-2)$ we have a smaller error term. Our assumption that $\rho > \frac{d-8}{2}$ is not imposed in [5], but this assumption is satisfied in all known applications. Our proof of Theorem 2.2 follows the method of [14]. It is completely different from the analysis used in [5], and is simpler technically and conceptually. By applying the fractional derivative method of [14, Section 2.3], the error term in (2.8) can be replaced by $O(\beta|||x|||^{-(d-2+s)})$ for any $s < \rho \wedge 2 \wedge (\rho - \frac{d-8}{2})$, with the constant independent of $z$, $\beta_0$, $\beta_1$, $L$. We omit such improvement as we do not need it for the proof of Theorem 1.7.

### 2.2. *Proof of Theorem 1.7: Bootstrap argument*

We now prove Theorem 1.7 assuming Proposition 1.2, Proposition 1.6, and Theorem 2.2. The following proposition is the core of the bootstrap argument.

**Proposition 2.3 (Bootstrap).** *Let $d > 2$. Under Assumptions 1.3 and 1.4, there is an $L_2$ such that for all $L \ge L_2$, if $z \in [1, z_c)$ and $b(z) \le 3$ then $b(z) \le 2$.*

**Proof.** Suppose $z \in [1, z_c)$ satisfies $b(z) \le 3$. By Assumptions 1.4, we have $F_z * G_z = \delta$ with $F_z = \delta - zD - \Pi_z$. We first verify Assumption 2.1 for $F_z$: (2.2) holds with $\beta = \beta_0$ proportional to $L^{-2+\varepsilon}$ and $\beta_1 = o(L^{-2+\varepsilon})$ by (1.14), and

$$\hat{F}_z(0) = \frac{1}{\hat{G}_z(0)} = \frac{1}{\sum_{x\in\mathbb{Z}^d} G_z(x)} \ge 0, \tag{2.10}$$

since $G_z$ is non-negative and summable when $z < z_c$.

We now use Theorem 2.2. Since $G_z$ satisfies $F_z * G_z = \delta$ and both $F_z$ and $G_z$ are summable, we have $\hat{F}_z(k)\hat{G}_z(k) = 1$ and hence $G_z$ is equal to the Fourier integral $\tilde{G}_z$ of (2.1). With $\lambda_z = 1 + O(\beta)$ from (2.6), Theorem 2.2 gives

$$G_z(x) = \big(1 + O(\beta)\big) S_{\mu_z}(x) + O\left(\frac{o(\beta)}{|x|^{d-2}}\right) \quad (x \neq 0), \tag{2.11}$$

where the constant is independent of $z$, $\beta$, $L$. By Proposition 1.2,

$$S_{\mu_z}(x) \le S_1(x) \le \frac{K_S}{L^{2-\varepsilon}|x|^{d-2}} \quad (x \neq 0). \tag{2.12}$$

Combined with $\beta = \text{const}\, L^{-2+\varepsilon}$, we get

$$G_z(x) \le \big(1 + O(\beta) + o(1)\big)\frac{K_S}{L^{2-\varepsilon}|x|^{d-2}} \le 2\frac{K_S}{L^{2-\varepsilon}|x|^{d-2}} \quad (x \neq 0), \tag{2.13}$$

for sufficiently large $L$ (we emphasise that $L$ is taken large here independently of $z$). Also, since $z = 1 + O(\beta)$, we have $3(z-1) = O(\beta) \le 2$. This proves that $b(z) \le 2$ for sufficiently large $L$. □

**Proof of Theorem 1.7.** By Proposition 1.6, it suffices to consider the impulse equation (1.15), so we work under Assumption 1.4.

By Proposition 2.3 and continuity of the function $b$, the interval $(2, 3]$ is forbidden for values of $b(z)$ when $z \in [1, z_c)$. Since $b(1) \le 1$ by Assumption 1.3(iv), we must have $b(z) \le 2$ for all $z \in [1, z_c)$. It then follows from Assumption 1.3(iii) that

$$G_{z_c}(x) = \lim_{z\to z_c^-} G_z(x) \le 2\frac{K_S}{L^{2-\varepsilon}|x|^{d-2}} \quad (x \neq 0). \tag{2.14}$$

The bound (2.14) implies that $b(z_c) \le 2$, so Assumption 1.4 gives a critical $F_{z_c} = \delta - z_c D - \Pi_{z_c}$ with $\Pi_{z_c}$ obeying (1.14). By monotone convergence, we can take the $z \to z_c^-$ limit of (2.10) to see that $\hat{F}_{z_c}(0) = 0$, so now Assumption 2.1 is verified at $z = z_c$.

For $z < z_c$, it follows from $b(z) \le 2$ and $G_z$ being summable that $G_z = \tilde{G}_z$, as in the proof of Proposition 2.3. To prove that also $G_{z_c} = \tilde{G}_{z_c}$, by the $L^2$ Fourier transform it suffices to show that $G_{z_c} \in \ell^2(\mathbb{Z}^d)$. This follows from $d > 4$, (2.14), and the fact that $G_{z_c}(0) < \infty$. The latter is a consequence of the $x = 0$ case of Theorem 2.2 applied to $G_z$ with $z < z_c$, together with (2.6) and the bound on $S_1(0)$ from (1.10):

$$G_{z_c}(0) = \lim_{z \to z_c^-} G_z(0) \le \big(1 + O(\beta)\big) S_1(0) + O(\beta) \le 1 + O(\beta). \tag{2.15}$$

Since $\hat{F}_{z_c}(0) = 0$, we see from (2.6) that $\mu_{z_c} = 1$. Therefore, by Theorem 2.2, Proposition 1.2, and (1.8), we have

$$G_{z_c}(x) \sim \lambda_{z_c} S_1(x) \sim \frac{\lambda_{z_c}}{\sigma^2} C_1(x) \sim \frac{\lambda_{z_c}}{\sigma^2} \frac{a_d}{|x|^{d-2}} \quad \text{as } |x| \to \infty, \tag{2.16}$$

which is the desired result. By (2.6), $\lambda_{z_c} = 1 + O(\beta)$, and $\lambda_{z_c}$ is given explicitly in terms of $\Pi_{z_c}$ as

$$\lambda_{z_c} = \frac{1}{z_c + \sigma^{-2} \sum_{x \in \mathbb{Z}^d} |x|^2 \Pi_{z_c}(x)}. \tag{2.17}$$

This completes the proof of Theorem 1.7, subject to Proposition 1.2, Proposition 1.6, and Theorem 2.2. □

**Remark 2.4.** The assumption that $d > 4$ in Theorem 1.7 is used only to justify that $G_{z_c}(x)$ is equal to the Fourier integral $\tilde{G}_{z_c}(x)$ of (2.1) (as mentioned in the previous proof, we know that $G_z = \tilde{G}_z$ for $z < z_c$). If we assume, in addition to Assumption 1.4, that $\Pi_z(x)$ is left-continuous at $z = z_c$ for all $x$, then we can relax to all $d > 2$, since then the equality $G_{z_c}(x) = \tilde{G}_{z_c}(x)$ follows from the infrared bound (2.3) together with the dominated convergence theorem to take the limit $z \to z_c^-$ in (2.1). This additional continuity assumption can be verified in practice (see [4, Appendix A]), but we do not comment further since all our applications have $d > 4$.

## 3. Proof of deconvolution Theorem 2.2

We follow the strategy of [14, Section 2.2], but additional care is required to track dependence on $\beta_0$, $\beta_1$, $L$.

### 3.1. *Fourier analysis*

For a function $g : \mathbb{T}^d \to \mathbb{C}$ and $p \in [1, \infty)$, we write

$$\|g\|_p^p = \int_{\mathbb{T}^d} \big|g(k)\big|^p \frac{dk}{(2\pi)^d} \tag{3.1}$$

for the $L^p(\mathbb{T}^d)$ norm, and we write $\|g\|_\infty$ for the supremum norm.

We first isolate the leading decay of $\tilde{G}_z$. Suppose $F_z$ obeys Assumption 2.1. For $\mu \in (0, 1]$, define $A_\mu = \delta - \mu D$, so that $A_\mu * S_\mu = \delta$ by (1.6). For any $\lambda \in \mathbb{R}$, we write

$$\begin{aligned} \tilde{G}_z &= \lambda S_\mu + \delta * \tilde{G}_z - \lambda S_\mu * \delta \\ &= \lambda S_\mu + (S_\mu * A_\mu) * \tilde{G}_z - \lambda S_\mu * (F_z * \tilde{G}_z) \\ &= \lambda S_\mu + S_\mu * E_{z,\lambda,\mu} * \tilde{G}_z, \end{aligned} \tag{3.2}$$

with

$$E_{z,\lambda,\mu} = A_\mu - \lambda F_z. \tag{3.3}$$

The choice of $\lambda_z$, $\mu_z$ in (2.4) has been made to ensure that

$$\sum_{x \in \mathbb{Z}^d} E_{z,\lambda_z,\mu_z}(x) = \sum_{x \in \mathbb{Z}^d} |x|^2 E_{z,\lambda_z,\mu_z}(x) = 0. \tag{3.4}$$

Indeed, (3.4) is a system of two linear equations in $\lambda$, $\mu$, with solution given by (2.4). This isolates the leading term $\lambda_z S_{\mu_z}$:

$$\tilde{G}_z = \lambda_z S_{\mu_z} + f_z, \qquad f_z = S_{\mu_z} * E_{z,\lambda_z,\mu_z} * \tilde{G}_z. \tag{3.5}$$

We only use (3.5) in its Fourier version, namely

$$\frac{1}{\hat{F}_z} = \frac{\lambda_z}{\hat{A}_\mu} + \hat{f}_z, \qquad \hat{f}_z = \frac{\hat{E}_{z,\lambda_z,\mu_z}}{\hat{A}_{\mu_z}\hat{F}_z}, \qquad \hat{E}_{z,\lambda_z,\mu_z} = \hat{A}_{\mu_z} - \lambda_z \hat{F}_z. \tag{3.6}$$

To simplify the notation, we will usually omit subscripts $z$, $\lambda_z$, $\mu_z$, and use subscripts to denote partial derivatives instead, e.g., $\hat{E}_\alpha = \nabla^\alpha \hat{E}_{z,\lambda_z,\mu_z}$ for a multi-index $\alpha$. The proof of Theorem 2.2 replies on a classical fact about the Fourier transform: smoothness of a function on $\mathbb{T}^d$ is related to the decay of its Fourier coefficient. Concretely, we have the following lemma, which repeats [14, Lemma 2.3] (an elementary proof is given in [14]). Necessary properties of weak derivatives are summarised in [14, Appendix A].

**Lemma 3.1.** *Let $a, d > 0$ be positive integers and let $h : \mathbb{Z}^d \to \mathbb{R}$. There is a constant $c_{d,a}$, depending only on the dimension $d$ and the maximal order $a$ of differentiation, such that if the weak derivative $\hat{h}_\alpha$ is in $L^1(\mathbb{T}^d)$ for all multi-indices $\alpha$ with $|\alpha| \le a$ then*

$$\big|h(x)\big| \le c_{d,a} \frac{1}{\|\!|x|\!\|^a} \max_{|\alpha|\in\{0,a\}} \|\hat{h}_\alpha\|_1. \tag{3.7}$$

*Moreover, $|x|^a h(x) \to 0$ as $|x| \to \infty$.*

Recall from (2.7) that, assuming $\rho > \frac{d-8}{2} \vee 0$,

$$n_d = \begin{cases} d-2 & \left(\rho \le 1 + \left(\frac{d-8}{2} \vee 0\right)\right), \\ d-1 & \left(\rho > 1 + \left(\frac{d-8}{2} \vee 0\right)\right). \end{cases} \tag{3.8}$$

For later reference, we observe that $n_d$ is the largest integer that satisfies

$$n_d < (d - 2 + \rho \wedge 2) \wedge \left(\frac{1}{2}d + 2 + \rho\right). \tag{3.9}$$

**Proposition 3.2.** *Let $F_z$ obey Assumption 2.1. Then the function $\hat{f}_z$ defined in (3.6) is $n_d$ times weakly differentiable, and for any multi-index $\alpha$ with $|\alpha| \le n_d$,*

$$\|\hat{f}_\alpha\|_r \lesssim \beta \qquad \left(r^{-1} > \frac{|\alpha| + 2 - \rho \wedge 2}{d}\right), \tag{3.10}$$

*with the constant independent of $z$, $\beta_0$, $\beta_1$, $L$. Moreover, if $|\alpha| \ne 0$, then*

$$\|\hat{f}_\alpha\|_r \lesssim \beta\big(L^{-c} + \beta\big) + \beta_1 \qquad \left(r^{-1} > \frac{|\alpha| + 2 - \rho \wedge 2}{d}\right) \tag{3.11}$$

*for some $c > 0$, with the constants independent of $z$, $\beta_0$, $\beta_1$, $L$.*

**Proof of Theorem 2.2 assuming Proposition 3.2.** By Proposition 3.2, $\hat{f}_z$ is $n_d$ times weakly differentiable, so $\nabla^\alpha \hat{f}_z \in L^1(\mathbb{T}^d)$ for all multi-indices $\alpha$ with $|\alpha| \le n_d$. By (3.9), $r = 1$ is permitted in (3.10) and (3.11) for all $|\alpha| \le n_d$. Since $\tilde{G}_z = \lambda_z S_{\mu_z} + f_z$, it follows from Lemma 3.1 and (3.10) that

$$\tilde{G}_z(x) = \lambda_z S_{\mu_z}(x) + O\left(\frac{\beta}{\|\!|x|\!\|^{n_d}}\right), \tag{3.12}$$

and that $f_z(x) = o(|x|^{-n_d})$ as $|x| \to \infty$ for fixed $z$, $\beta_0$, $\beta_1$, $L$. For the improved dependence of the error term on $\beta$, $L$ when $x \ne 0$, we note that the $|\alpha| = 0$ part of (3.7) is only used to estimate $h(0)$. Therefore, it suffices to observe that (3.11) implies that $\|\hat{f}_\alpha\|_1 \lesssim \beta(L^{-c} + \beta) + \beta_1$ for some $c > 0$ for all $|\alpha| = n_d$. □

The proof of Proposition 3.2 uses product and quotient rules of differentiation. Since $\hat{f} = \hat{E}/(\hat{A}\hat{F})$, the $\alpha$-th weak derivative of $\hat{f}$ is given by a linear combination of terms of the form

$$\frac{\prod_{n=1}^{i} \hat{A}_{\delta_n}}{\hat{A}^{1+i}} \hat{E}_{\alpha_2} \frac{\prod_{m=1}^{j} \hat{F}_{\gamma_m}}{\hat{F}^{1+j}} = \left( \prod_{n=1}^{i} \frac{\hat{A}_{\delta_n}}{\hat{A}} \right) \left( \frac{\hat{E}_{\alpha_2}}{\hat{A}\hat{F}} \right) \left( \prod_{m=1}^{j} \frac{\hat{F}_{\gamma_m}}{\hat{F}} \right), \tag{3.13}$$

where $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, $0 \le i \le |\alpha_1|$, $0 \le j \le |\alpha_3|$, $\sum_{n=1}^{i} \delta_n = \alpha_1$, and $\sum_{m=1}^{j} \gamma_m = \alpha_3$, provided that we can justify *a posteriori* that all terms of the form (3.13) are integrable (see [14, Appendix A]). For this, we use Hölder's inequality and the following lemma.

**Lemma 3.3.** *Let $F_z$ obey Assumption 2.1. Let $\gamma$ be a multi-index with $|\gamma| < (d - 2 + \rho \wedge 2) \wedge (\frac{1}{2}d + 2 + \rho)$. Choose $\sigma \in (0, \rho \wedge 2)$ and $q_1, q_2$ satisfying*

$$\frac{|\gamma|}{d} < q_1^{-1} < 1, \qquad \frac{2 - \sigma + |\gamma|}{d} < q_2^{-1} < 1. \tag{3.14}$$

*Then $\hat{F}$, $\hat{A}$, $\hat{E}$ are $\gamma$-times weakly differentiable, and*

$$\left\| \frac{\hat{A}_\gamma}{\hat{A}} \right\|_{q_1}, \left\| \frac{\hat{F}_\gamma}{\hat{F}} \right\|_{q_1} \lesssim 1, \qquad \left\| \frac{\hat{E}_\gamma}{\hat{A}\hat{F}} \right\|_{q_2} \lesssim \beta, \tag{3.15}$$

*with the constants independent of $z$, $\beta_0$, $\beta_1$, $L$. Moreover, if $|\gamma| \neq 0$, the estimates are improved to*

$$\left\| \frac{\hat{A}_\gamma}{\hat{A}} \right\|_{q_1} \lesssim L^{|\gamma| - d/q_1}, \qquad \left\| \frac{\hat{F}_\gamma}{\hat{F}} \right\|_{q_1} \lesssim L^{|\gamma| - d/q_1} + \beta, \qquad \left\| \frac{\hat{E}_\gamma}{\hat{A}\hat{F}} \right\|_{q_2} \lesssim \beta L^{2 - \sigma + |\gamma| - d/q_2} + \beta_1, \tag{3.16}$$

*with the constants independent of $z$, $\beta_0$, $\beta_1$, $L$.*

**Proof of Proposition 3.2 assuming Lemma 3.3.** Let $|\alpha| \le n_d$, $\rho_2 = \rho \wedge 2$, and pick some $\sigma \in (0, \rho_2)$. We use the product and quotient rules of weak derivatives [14, Lemmas A.2–A.3] to calculate $\hat{f}_\alpha$. For the hypotheses of these rules, we need to verify all terms of the form (3.13) are integrable. By Hölder's inequality and Lemma 3.3, (3.13) belongs to $L^r(\mathbb{T}^d)$ as long as

$$\frac{1}{r} > \frac{\sum_{n=1}^{i} |\delta_n|}{d} + \frac{2 - \sigma + |\alpha_2|}{d} + \frac{\sum_{m=1}^{j} |\gamma_m|}{d} = \frac{|\alpha| + 2 - \sigma}{d}. \tag{3.17}$$

Since $\sigma < \rho_2$ is arbitrary, this shows that (3.13) is in $L^r$ for all $r^{-1} > (|\alpha| + 2 - \rho_2)/d$. In particular, it belongs to $L^1$ since $|\alpha| \le n_d < d - 2 + \rho_2$ by (3.9). This proves that $\hat{f}$ is $\alpha$-times weakly differentiable and that $\hat{f}_\alpha \in L^r$ with the same values of $r$. Furthermore, we get a quantitative estimate on $\| \hat{f}_\alpha \|_r$ from Hölder's inequality. For (3.10), we use (3.15) and get $\beta$ from the norm of $\hat{E}_{\alpha_2}/(\hat{A}\hat{F})$. For (3.11), since there is at least one derivative taken, in one of the factors we can use the stronger (3.16). The constant $c > 0$ is produced by the strict inequalities in (3.14). □

To complete the proof of Theorem 2.2, it remains to prove Lemma 3.3. The proof uses the following elementary facts about the Fourier transform. The first lemma translates the good moment behaviour of $E(x)$ in (3.4) (the first moments of $E(x)$ also vanish, by symmetry) into good bounds on $\hat{E}(k)$ and its derivatives, which ultimately allows us to take $n_d$ derivatives of $\hat{f}$. The second lemma uses boundedness of the $L^p$ Fourier transform when $1 \le p \le 2$.

**Lemma 3.4 ([14, Lemma 2.2]).** *Suppose $E : \mathbb{Z}^d \to \mathbb{R}$ is $\mathbb{Z}^d$-symmetric, has vanishing zeroth and second moments as in (3.4), and satisfies $|E(x)| \le K|x|^{-(d+2+\rho)}$ for some $K, \rho > 0$. Choose $\sigma \in (0, \rho)$ such that $\sigma \le 2$ and let $\alpha$ be a multi-index with $|\alpha| < 2 + \sigma$. Then there is a constant $c = c(\sigma, \rho, d)$ such that*

$$\left| \hat{E}_\alpha(k) \right| \le cK \cdot |k|^{2 + \sigma - |\alpha|}. \tag{3.18}$$

**Lemma 3.5 ([14, Lemma 2.6]).** *Let $h : \mathbb{Z}^d \to \mathbb{R}$ obey $|h(x)| \le K \|\!|x|\!\|^{-b}$ for some $K, b > 0$.*

(i) *If $b > d$ then $h \in \ell^1(\mathbb{Z}^d)$, $\hat{h} \in L^\infty(\mathbb{T}^d)$, and $\|\hat{h}\|_\infty \le c_{d,b} K$.*
(ii) *If $b \le d$ then $h \in \ell^p(\mathbb{Z}^d)$ for $p > d/b$. If also $\frac{d}{2} < b \le d$ then $\hat{h} \in L^q(\mathbb{T}^d)$ and $\|\hat{h}\|_q \le c_{d,b,q} K$ for all $1 \le q < d/(d-b)$.*

## 3.2. *Proof of Lemma* 3.3

We first collect properties of $D$ that we need. The proof of Lemma 3.6 is deferred to Appendix B.

**Lemma 3.6.** *If* $L \geq L_0$ *with* $L_0$ *sufficiently large (depending only on* $d, v$*), then the following statements hold. For any* $a > 0$,

$$D(x) \lesssim \frac{L^a}{|\!|\!| x |\!|\!|^{d+a}}. \tag{3.19}$$

*Uniformly in* $\mu \in [\frac{1}{2}, 1]$, $\hat{A}_\mu = 1 - \mu \hat{D}$ *satisfies the* infrared bound

$$\hat{A}_\mu(k) - \hat{A}_\mu(0) \gtrsim L^2|k|^2 \wedge 1 \quad \big(k \in \mathbb{T}^d\big). \tag{3.20}$$

*For each multi-index* $\alpha$,

$$\|\hat{D}_\alpha\|_q \lesssim L^{|\alpha|-d/q} \quad \big(0 \leq q^{-1} < 1\big). \tag{3.21}$$

Together with (3.20), Assumption 2.1 implies an infrared bound for $\hat{F}_z$. To see this, we first write

$$\hat{F}_z(k) - \hat{F}_z(0) = z\big(1 - \hat{D}(k)\big) + \big(\hat{\Pi}_z(0) - \hat{\Pi}_z(k)\big). \tag{3.22}$$

The first term is bounded from below using $z \geq 1$ and (3.20), and the second term is bounded in absolute value by $O(\beta)(|k|^2 \wedge 1)$, by Taylor's theorem, symmetry, and (2.2). Since $\beta$ is small, this yields

$$\hat{F}_z(k) - \hat{F}_z(0) \geq K_{\mathrm{IR}}\big(L^2|k|^2 \wedge 1\big) \quad \big(k \in \mathbb{T}^d\big) \tag{3.23}$$

for some $K_{\mathrm{IR}} > 0$. Because of the two alternatives in the infrared bounds (3.20) and (3.23), we pay separate attention to the small ball

$$B_L = \big\{k \in \mathbb{R}^d : \|k\|_\infty \leq \pi,\ |k| < 1/L\big\}, \tag{3.24}$$

and to its complement.

**Proof of Lemma 3.3.** *Bound on* $\hat{A}_\gamma/\hat{A}$. There is nothing to prove for $|\gamma| = 0$ since the ratio is then 1. We will prove that, for $|\gamma| \geq 1$,

$$\left\| \frac{\hat{A}_\gamma}{\hat{A}} \right\|_q \lesssim L^{|\gamma|-d/q} \quad \left(\frac{|\gamma| \wedge 2}{d} < q^{-1} < 1\right). \tag{3.25}$$

This is stronger than the desired (3.16) by allowing more values of $q = q_1$. It also implies $\|\hat{A}_\gamma/\hat{A}\|_q \lesssim 1$ when we restrict to $|\gamma|/d < q^{-1} < 1$. By the infrared bound (3.20),

$$\left| \frac{\hat{A}_\gamma}{\hat{A}}(k) \right| \lesssim L^{-2}|k|^{-2}\big|\hat{A}_\gamma(k)\big| \mathbb{1}_{B_L} + \big|\hat{A}_\gamma(k)\big|, \tag{3.26}$$

where $B_L$ is the small ball in (3.24). Since $\hat{A}_\gamma = -\mu \hat{D}_\gamma$, by (3.21) the $L^q$ norm of the second term on the right-hand side is bounded by $L^{|\gamma|-d/q}$, as required. If $|\gamma| = 1$, Taylor's Theorem and symmetry give $|\hat{A}_\gamma(k)| \lesssim L^2|k|$, so the first term is bounded by $|k|^{-1}\mathbb{1}_{B_L}$, which has $L^q$ norm bounded by $L^{1-d/q}$ for $q^{-1} > 1/d$. For the remaining case $|\gamma| \geq 2$, it follows from Hölder's inequality and (3.21) that

$$\left\| \frac{\hat{A}_\gamma}{\hat{A}} \mathbb{1}_{B_L} \right\|_q \lesssim \frac{1}{L^2} \|\hat{A}_\gamma\|_r \big\| |k|^{-2} \mathbb{1}_{B_L} \big\|_p \lesssim \frac{1}{L^2} \big(L^{|\gamma|-d/r}\big) L^{2-d/p} = L^{|\gamma|-d/q} \tag{3.27}$$

for $q^{-1} = r^{-1} + p^{-1}$ with $0 \leq r^{-1} < 1$ and $p^{-1} > 2/d$, which in particular holds for any $q^{-1} > 2/d$. This completes the proof of (3.25).

*Bound on* $\hat{F}_\gamma/\hat{F}$. The $|\gamma|=0$ case is again trivial. We will prove that if $1 \le |\gamma| < \frac{1}{2}d+2+\rho$ then

$$\left\| \frac{\hat{F}_\gamma}{\hat{F}} \right\|_q \lesssim L^{|\gamma|-d/q} + \beta \lesssim 1 \quad \left( \frac{|\gamma|}{d} < q^{-1} < 1 \right). \tag{3.28}$$

The second inequality holds because $\beta$ is small. As in (3.26), by the infrared bound (3.23),

$$\left| \frac{\hat{F}_\gamma}{\hat{F}}(k) \right| \lesssim L^{-2}|k|^{-2}\big|\hat{F}_\gamma(k)\big|\mathbb{1}_{B_L} + \big|\hat{F}_\gamma(k)\big|. \tag{3.29}$$

Since $\hat{F} = 1 - z\hat{D} - \hat{\Pi}$ and $|\gamma| \ge 1$, by the triangle inequality, by the fact that $z \le O(1)$ by (2.5), by (3.21), and by the Fourier transform bound Lemma 3.5(ii) applied with $h(x) = i^{|\gamma|}x^\gamma \Pi(x)$,

$$\|\hat{F}_\gamma\|_q \le |z|\|\hat{D}_\gamma\|_q + \|\hat{\Pi}_\gamma\|_q \lesssim L^{|\gamma|-d/q} + \beta \quad \left( 0 \le q^{-1} < 1, q^{-1} > \frac{|\gamma|-2-\rho}{d} \right). \tag{3.30}$$

This gives the desired bound for the second term of (3.29). If $|\gamma|=1$, Taylor's Theorem and symmetry give $|\hat{F}_\gamma(k)| \lesssim (L^2+\beta)|k|$, so the first term is bounded by $(1+\beta/L^2)|k|^{-1}\mathbb{1}_{B_L}$, which has $L^q$ norm bounded by $(1+\beta/L^2)L^{1-d/q} \lesssim L^{1-d/q}$ when $q^{-1} > 1/d$. For the remaining case $|\gamma| \ge 2$, we let $r^{-1} = (|\gamma|-2)/d$. It follows from Hölder's inequality and (3.30) that

$$\left\| \frac{\hat{F}_\gamma}{\hat{F}}\mathbb{1}_{B_L} \right\|_q \lesssim \frac{1}{L^2}\|\hat{F}_\gamma\|_r \big\| |k|^{-2}\mathbb{1}_{B_L} \big\|_p \lesssim \frac{1}{L^2}\big(L^{|\gamma|-d/r}+\beta\big)L^{2-d/p} \lesssim L^{|\gamma|-d/q} \tag{3.31}$$

for $q^{-1} = r^{-1} + p^{-1}$ and $p^{-1} > 2/d$. By the choice of $r$, the bound holds for $q^{-1} > |\gamma|/d$. This completes the proof of (3.28).

*Bound on* $\hat{E}_\gamma/\hat{A}\hat{F}$. Let $|\gamma| < \frac{1}{2}d+2+\rho$ and choose $0 < \sigma < \rho \wedge 2$. Our goal is to prove that

$$\left\| \frac{\hat{E}_\gamma}{\hat{A}\hat{F}} \right\|_q \lesssim \beta \quad \left( \frac{2-\sigma+|\gamma|}{d} < q^{-1} < 1 \right), \tag{3.32}$$

which establishes (3.15), and to improve the bound when $|\gamma| \ne 0$ to

$$\left\| \frac{\hat{E}_\gamma}{\hat{A}\hat{F}} \right\|_q \lesssim \beta L^{2-\sigma+|\gamma|-d/q} + \beta_1 \quad \left( \frac{2-\sigma+|\gamma|}{d} < q^{-1} < 1 \right), \tag{3.33}$$

which is (3.16).

It follows from the formula $\hat{E} = \hat{A}_\mu - \lambda\hat{F}$ in (3.6), together with the fact that $\mu = 1 - \lambda\hat{F}(0)$ by (2.4), that

$$\hat{E} = (1-\lambda)(1-\hat{D}) - \lambda\hat{\Pi}(0)\hat{D} + \lambda\hat{\Pi}. \tag{3.34}$$

Also, by the infrared bounds for $\hat{A}$ and $\hat{F}$,

$$\left| \frac{\hat{E}_\gamma}{\hat{A}\hat{F}}(k) \right| \lesssim L^{-4}|k|^{-4}\big|\hat{E}_\gamma(k)\big|\mathbb{1}_{B_L} + \big|\hat{E}_\gamma(k)\big|. \tag{3.35}$$

We will first show that the second term on the right-hand side of (3.35) obeys (3.32) and (3.33), and then show that the first term on the right-hand side of (3.35) obeys the stronger bound (3.33) even when $|\gamma|=0$.

For the second term on the right-hand side of (3.35), we use (3.34), the triangle inequality, $\lambda = 1 + O(\beta)$ (by (2.6)), and $|\hat{\Pi}(0)| \lesssim \beta$, to see that

$$\|\hat{E}_\gamma\|_q \lesssim O(\beta)\big(1 + \|\hat{D}_\gamma\|_q\big) + |\lambda|\|\hat{\Pi}_\gamma\|_q. \tag{3.36}$$

By (3.21), $\|\hat{D}_\gamma\|_q \lesssim L^{|\gamma|-d/q}$. We bound the norm of $\hat{\Pi}_\gamma$ using (2.2) and Lemma 3.5, to see that

$$\|\hat{E}_\gamma\|_q \lesssim \beta\big(1 + L^{|\gamma|-d/q}\big) \quad \left( 0 \le q^{-1} < 1,\ q^{-1} > \frac{|\gamma|-2-\rho}{d} \right). \tag{3.37}$$

In particular, (3.37) holds for $(2-\sigma+|\gamma|)/d < q^{-1} < 1$, and the norm is bounded by a multiple of $\beta$ since $\sigma \le 2$. To improve the bound when $|\gamma| \neq 0$, we still use (3.34), but now the contribution from the term $(1-\lambda)(1)$ vanishes because there is at least one derivative taken. For the same reason, we can write $\hat{\Pi}_\gamma = \nabla^\gamma(\hat{\Pi} - \Pi(0))$. Since $\hat{\Pi} - \Pi(0)$ is the Fourier transform of $\Pi(x) - \Pi(0)\delta_{0,x}$, Lemma 3.5 and the $x \neq 0$ part of (2.2) imply that

$$\|\hat{E}_\gamma\|_q \lesssim \beta \|\hat{D}_\gamma\|_q + \|\hat{\Pi}_\gamma\|_q \lesssim \beta L^{|\gamma|-d/q} + \beta_1 \quad \left(0 \le q^{-1} < 1,\ q^{-1} > \frac{|\gamma|-2-\rho}{d}\right), \tag{3.38}$$

which implies that $|\hat{E}_\gamma|$ obeys the upper bound in (3.33), since $\sigma < \rho$ and $\sigma \le 2$.

For the first term on the right-hand side of (3.35), we will prove that the stronger bound (3.33) holds for all $\gamma$. Consider first the case $|\gamma| < 2+\sigma$. It follows from the $x$-space version of (3.34), together with the decay of $D$ in (3.19) and of $\Pi$ in (2.2) that

$$\left|E(x)\right| \lesssim \frac{\beta L^{2+\rho_2}}{\|\!|x|\!\|^{d+2+\rho_2}} \tag{3.39}$$

(we have relaxed the decay of $\Pi$ because it is costly to make $D$ decay). Then Lemma 3.4 with $\rho_2$ in place of $\rho$ gives (it is here that we require the strict inequality $\sigma < \rho_2$)

$$\left|\hat{E}_\gamma(k)\right| \lesssim \beta L^{2+\rho_2} |k|^{2+\sigma-|\gamma|}. \tag{3.40}$$

The $L^q$ norm of $|k|^{2+\sigma-|\gamma|-4} = |k|^{-(2-\sigma+|\gamma|)}$ on $B_L$ is of order $L^{2-\sigma+|\gamma|-d/q}$, so the $L^q$ norm of the first term on on the right-hand side of (3.35) is bounded, in this case, by the desired

$$\frac{\beta L^{2+\rho_2}}{L^4} L^{2-\sigma+|\gamma|-d/q} \le \beta L^{2-\sigma+|\gamma|-d/q} \le \beta \quad \left(q^{-1} > \frac{2-\sigma+|\gamma|}{d}\right). \tag{3.41}$$

For the remaining case $2+\sigma \le |\gamma| < \frac{1}{2}d + 2 + \rho$, Hölder's inequality and (3.37) imply that the $L^q$ norm of the first term on the right-hand side of (3.35) is bounded above by

$$\frac{1}{L^4} \|\hat{E}_\gamma\|_r \left\| |k|^{-4} \mathbb{1}_{B_L} \right\|_p \lesssim \frac{1}{L^4} \left[\beta\left(1 + L^{|\gamma|-d/r}\right)\right] L^{4-d/p} \lesssim \beta L^{|\gamma|-d/q} \tag{3.42}$$

for $q^{-1} = r^{-1} + p^{-1}$, $r \in (1,\infty]$, $\frac{|\gamma|-2-\rho}{d} < r^{-1} \le \frac{|\gamma|}{d}$, and $p^{-1} > 4/d$. In particular, since $\sigma < \rho$, the bound holds for all $q^{-1} > (2-\sigma+|\gamma|)/d$. The desired bound (3.33) for the first term on the right-hand side of (3.35) then follows from $\sigma \le 2$. This completes the proof of (3.32) and concludes the proof of the lemma. ∎

## 4. Inhomogeneous deconvolution: Proof of Proposition 1.6

We now prove Proposition 1.6, which concerns the inhomogeneous convolution equation

$$G_z = h_z + zD * h_z * G_z. \tag{4.1}$$

In fact we prove a stronger proposition, with arbitrary small $\beta_0$ and $\beta_1$ rather than the specific choices in (1.18). We write $\theta = 2 + \rho$.

**Proposition 4.1.** *Suppose the function $h_z : \mathbb{Z}^d \to \mathbb{R}$ is $\mathbb{Z}^d$-symmetric and satisfies*

$$\left|h_z(x) - \delta_{0,x}\right| \le \beta_0 \delta_{0,x} + \frac{\beta_1}{\|\!|x|\!\|^{d+\theta}} \tag{4.2}$$

*with $\theta > 0$ and with $\beta = \beta_0 \vee \beta_1 \ge 0$ sufficiently small. Then there exists a $\mathbb{Z}^d$-symmetric function $\Phi_z : \mathbb{Z}^d \to \mathbb{R}$ for which $G_z$ of (4.1) satisfies $F_z * G_z = \delta$ with*

$$F_z = \delta - zD - \Phi_z, \qquad \left|\Phi_z(x)\right| \le O(\beta)\delta_{0,x} + \frac{O(\beta_1)}{\|\!|x|\!\|^{d+\theta}}. \tag{4.3}$$

The proof uses a Banach algebra, as in [1]. Given $\zeta > 0$, we define a norm on functions $v : \mathbb{Z}^d \to \mathbb{R}$ by

$$\|v\|_\zeta = \max\Big\{2^{\zeta+1}\sum_{x\in\mathbb{Z}^d}|v(x)|,\ \sup_{x\in\mathbb{Z}^d}|x|^\zeta|v(x)|\Big\}. \tag{4.4}$$

Then $\|u * v\|_\zeta \le \|u\|_\zeta\|v\|_\zeta$ for all $u$ and $v$, so the space $\{v : \|v\|_\zeta < \infty\}$ is a Banach algebra with product given by convolution. If $\|v - \delta\|_\zeta < 1$, then $v$ has a deconvolution $v^{-1} = \sum_{n=0}^\infty(\delta - v)^{*n}$ given by a convergent Neumann series. Indeed, by writing $v = \delta - (\delta - v)$, we have

$$v * v^{-1} = \sum_{n=0}^\infty(\delta - v)^{*n} - \sum_{n=1}^\infty(\delta - v)^{*n} = \delta. \tag{4.5}$$

Also, $v^{-1}$ satisfies

$$\big\|v^{-1} - \delta\big\|_\zeta \le \sum_{n=1}^\infty \|\delta - v\|_\zeta^n = \frac{\|v-\delta\|_\zeta}{1 - \|v-\delta\|_\zeta}. \tag{4.6}$$

**Proof of Proposition 4.1.** We drop subscripts $z$ from the notation as they play no role in the proof. By (4.2), we have $\|h-\delta\|_\zeta \le O(\beta) < 1$ with $\zeta = d+\theta$, so the deconvolution $h^{-1}$ exists and (4.6) holds with $v = h$. Define $F = \delta - zD - \Phi$ with $\Phi = \delta - h^{-1}$. Then

$$F = (\delta - \Phi) - zD = h^{-1} - zD = h^{-1} * (\delta - zD * h), \tag{4.7}$$

so that, using (4.1) in the second equality,

$$F * G = h^{-1} * (G - zD * h * G) = h^{-1} * h = \delta. \tag{4.8}$$

For the decay of $\Phi$, we first use $\|h - \delta\|_\zeta \le O(\beta)$, (4.6), and the definition of $\|\cdot\|_\zeta$ to get

$$\max\Big\{2^{\zeta+1}|\Phi(0)|,\ \sup_{x\in\mathbb{Z}^d}|x|^\zeta|\Phi(x)|\Big\} \le \|\Phi\|_\zeta = \big\|h^{-1} - \delta\big\|_\zeta \le \frac{O(\beta)}{1 - O(\beta)}. \tag{4.9}$$

This proves that $|\Phi(x)| \le O(\beta)|\!|\!|x|\!|\!|^{-\zeta}$ and gives the bound on $\Phi(0)$ in (4.3), but for $x \ne 0$ we wish to improve the $O(\beta)$ to $O(\beta_1)$. We make the improvement as follows. Let $f = \delta - h$, so that

$$\Phi = \delta - h^{-1} = -\sum_{n=1}^\infty(\delta - h)^{*n} = -\sum_{n=1}^\infty f^{*n}. \tag{4.10}$$

For $x \ne 0$, we have $\Phi(x) = \Phi(x) - \Phi(0)\delta(x)$. With $g_n = f^{*n} - f^{*n}(0)\delta$, we write this as

$$\Phi(x) - \Phi(0)\delta(x) = -\sum_{n=1}^\infty g_n(x) \quad (x \ne 0). \tag{4.11}$$

By definition, for $n \ge 1$,

$$g_{n+1}(x) = (g_n * f)(x) + f^{*n}(0)g_1(x) \quad (x \ne 0). \tag{4.12}$$

Since $g_{n+1}(0) = 0$ and $|f^{*n}(0)| \le \|f^{*n}\|_\zeta \le \|f\|_\zeta^n$, it follows from the triangle inequality that

$$\|g_{n+1}\|_\zeta \le \|f\|_\zeta\|g_n\|_\zeta + \|f\|_\zeta^n\|g_1\|_\zeta. \tag{4.13}$$

It then follows by iteration that

$$\|g_n\|_\zeta \le n\|f\|_\zeta^{n-1}\|g_1\|_\zeta \quad (n \ge 1). \tag{4.14}$$

Since we have $\|f\|_\zeta \le O(\beta)$ and $\|g_1\|_\zeta \le O(\beta_1)$ by the definition of $f$ and (4.2), we can bound the norm of the sum in (4.11) as

$$\big\|\Phi - \Phi(0)\delta\big\|_\zeta \le O(\beta_1), \tag{4.15}$$

which implies the desired bound $|\Phi(x)| \le O(\beta_1)|x|^{-\zeta}$ for $x \ne 0$. This concludes the proof. □

For lattice trees and lattice animals, we cannot verify (4.2) directly, because their one-point function plays a role without counterpart in the other models. However, only a small adjustment is needed to apply our results. The lace expansion for the two-point function $T_p$ of these models has the form

$$T_p = t_p + pD * t_p * T_p. \tag{4.16}$$

We divide out the one-point function as in [5, (1.23)]. We set $\tau_p = t_p(0)$, $z = p\tau_p$, $h_z = t_p/\tau_p$, and $G_z = T_p/\tau_p$. This transforms the above equation to (4.1) with $h_z$ now a small perturbation of $\delta$, and Proposition 4.1 can be applied.

## Appendix A: Spread-out Green function

We now prove Proposition 1.2, which asserts that if $D$ is given by Definition 1.1, if $d > 2$, if $L \ge L_0$, and if $\varepsilon > 0$, then the critical spread-out Green function $S_1(x)$ satisfies

$$S_1(x) = \delta_{0,x} + \frac{1}{\sigma^2} C_1(x) + O\left(\frac{1}{L^{1-\varepsilon} |||x|||^{d-1}}\right) \tag{A.1}$$

and $S_1(x) \le \delta_{0,x} + K_S L^{-(2-\varepsilon)} |||x|||^{-(d-2)}$ for some $K_S = K_S(\varepsilon)$, with constants uniform in $L$ but dependent on $\varepsilon$. Our proof uses the same steps used to prove Theorem 2.2, and is conceptually and technically simpler than the proof using intricate Fourier analysis in [5]. We use the conclusion of Lemma 3.6 repeatedly. Although Lemma 3.6 is proved later, there is no circularity because the proof of Lemma 3.6 is independent of the proofs here.

We isolate the leading term as follows. Recall that $P(x) = \frac{1}{2d} \mathbb{1}_{|x|=1}$ and that $\sigma^2$ is the variance of $D$. Let $A = \delta - P$, $F = \delta - D$, $E = A - \sigma^{-2} F$, and $f = C_1 * E * S_1$. Then a similar calculation as in (3.2), with $\lambda = \sigma^{-2}$, with $S_\mu$ replaced by $C_1$, and with $G_z$ replaced by $S_1$, gives

$$S_1 = \sigma^{-2} C_1 + f, \tag{A.2}$$

with $E(x)$ having vanishing zeroth and second moments as in (3.4). We further extract a Kronecker delta using $(\delta - D) * S_1 = \delta$, to get

$$S_1 = \delta + D * S_1 = \delta + D * \left(\sigma^{-2} C_1 + f\right) = \delta + \sigma^{-2} C_1 + \varphi, \tag{A.3}$$

where

$$\varphi = D * f - \sigma^{-2} C_1 * F. \tag{A.4}$$

Now the error term $\varphi$ plays the role played by $f$ in Section 3.

**Lemma A.1.** *Let $\varepsilon > 0$. If $L \ge L_0$ with $L_0$ sufficiently large (depending only on $d$, $v$), then the following statements hold. The function $\hat{\varphi}$ is $d-1$ times weakly differentiable, and for any multi-index $\alpha$ with $|\alpha| \le d-1$,*

$$\|\hat{\varphi}_\alpha\|_1 \lesssim L^{-(1-\varepsilon)}, \tag{A.5}$$

*with the constant independent of $L$ but depends on $\varepsilon$.*

**Proof of Proposition 1.2 assuming Lemma A.1.** It follows from Lemma 3.1 and (A.5) that $\varphi(x) = O(L^{-(1-\varepsilon)} \times |||x|||^{-(d-1)})$, which implies (A.1). Also, the bound $S_1(x) \le \delta_{0,x} + K_S L^{-(2-\varepsilon)} |||x|||^{-(d-2)}$ follows from (A.1) and the asymptotic formula for $C_1(x)$ in (1.8) when $|x| > L$, and follows from $S_1(x) = \delta_{0,x} + O(L^{-d})$ when $|x| \le L$ (this simple fact is proved in [5, Section 6.1]). □

To prove Lemma A.1, we use the following analogue of Lemma 3.3. Its proof involves only a minor adaptation of the proof of Lemma 3.3. (We write $\tau$ in place of the $\sigma$ in Lemma 3.3 because here we reserve $\sigma^2$ for the variance of $D$.)

**Lemma A.2.** *Let $L \ge L_0$ with $L_0$ sufficiently large (depending only on $d$, $v$). Let $\gamma$ be a multi-index with $|\gamma| < d$. Choose $\tau \in (0,2)$ and $q_1$, $q_2$ satisfying*

$$\frac{|\gamma|}{d} < q_1^{-1} < 1, \qquad \frac{2 - \tau + |\gamma|}{d} < q_2^{-1} < 1. \tag{A.6}$$

*Then $\hat F$, $\hat A$, $\hat E$ are $\gamma$-times weakly differentiable and*

$$\left\| \frac{\hat A_\gamma}{\hat A} \right\|_{q_1}, \left\| \frac{\hat F_\gamma}{\hat F} \right\|_{q_1}, \left\| \frac{\hat E_\gamma}{\hat A \hat F} \right\|_{q_2} \lesssim 1, \tag{A.7}$$

*with constants independent of $L$.*

Given Lemma A.2, it follows exactly as in the proof of Proposition 3.2 that the function $\hat f = \hat E/(\hat A \hat F)$ is $d-1$ times weakly differentiable, and

$$\|\hat f_\alpha\|_r \lesssim 1 \quad \left( r^{-1} > \frac{|\alpha|}{d} \right) \tag{A.8}$$

for $|\alpha| \le d-1$, with the constant independent of $L$. We use this to prove Lemma A.1 first, and then we complete the proof of Proposition 1.2 by proving Lemma A.2.

**Proof of Lemma A.1 assuming Lemma A.2.** For simplicity, we write $C = C_1$ and $\hat C = 1/\hat A$. It suffices to consider small $\varepsilon > 0$. By the product rule, and since $\sigma^{-2} \lesssim L^{-2}$, it suffices to prove that, for $|\alpha_1| + |\alpha_2| = |\alpha| \le d-1$,

$$\|\hat f_{\alpha_1} \hat D_{\alpha_2}\|_1 \lesssim L^{-(1-\varepsilon)}, \tag{A.9}$$

$$\|\hat C_{\alpha_1} \hat F_{\alpha_2}\|_1 \lesssim L^{1+\varepsilon}. \tag{A.10}$$

The bound (A.9) directly follows from Hölder's inequality, (A.8), and (3.21):

$$\|\hat f_{\alpha_1} \hat D_{\alpha_2}\|_1 \le \|\hat f_{\alpha_1}\|_{\frac{d}{|\alpha_1|+\varepsilon}} \|\hat D_{\alpha_2}\|_{\frac{d}{|\alpha_2|+1-\varepsilon}} \lesssim L^{|\alpha_2|-(|\alpha_2|+1-\varepsilon)} = L^{-(1-\varepsilon)}. \tag{A.11}$$

To prove (A.10), we first note that, by a direct computation using the explicit formula

$$\hat C_1(k) = \frac{1}{\hat A(k)} = \frac{1}{1-\hat P(k)}, \qquad P(k) = d^{-1} \sum_{j=1}^d \cos k_j, \tag{A.12}$$

we have $|\hat C_{\alpha_1}(k)| \lesssim |k|^{-(2+|\alpha_1|)}$ for all $\alpha_1$. Consider first the case $|\alpha_2| < 1+\varepsilon$. Since $\sum_{x\in\mathbb{Z}^d} F(x) = 0$ and $1+\varepsilon \le 2$, Taylor expansion at $k=0$ gives

$$\left|\hat F_{\alpha_2}(k)\right| \le \sum_{x\in\mathbb{Z}^d} \left|F(x)\right| \left|\nabla^{\alpha_2}\big(\cos(k\cdot x) - 1\big)\right| \lesssim \sum_{x\in\mathbb{Z}^d} \left|F(x)\right| |k|^{1+\varepsilon-|\alpha_2|} |x|^{1+\varepsilon} \lesssim L^{1+\varepsilon} |k|^{1+\varepsilon-|\alpha_2|}, \tag{A.13}$$

so that

$$\|\hat C_{\alpha_1} \hat F_{\alpha_2}\|_1 \lesssim L^{1+\varepsilon} \left\| |k|^{-(2+|\alpha_1|+|\alpha_2|-1-\varepsilon)} \right\|_1 \lesssim L^{1+\varepsilon}, \tag{A.14}$$

since $2 + |\alpha_1| + |\alpha_2| - 1 - \varepsilon \le d - \varepsilon < d$. If instead $|\alpha_2| \ge 1+\varepsilon$, then we use Hölder's inequality and the norm bound (3.21) on $\hat F = 1 - \hat D$ to complete the proof of (A.10) with

$$\|\hat C_{\alpha_1} \hat F_{\alpha_2}\|_1 \le \|\hat C_{\alpha_1}\|_{\frac{d}{2+|\alpha_1|+\varepsilon}} \|\hat F_{\alpha_2}\|_{\frac{d}{|\alpha_2|-1-\varepsilon}} \lesssim L^{|\alpha_2|-(|\alpha_2|-1-\varepsilon)} = L^{1+\varepsilon}. \tag{A.15}$$

This concludes the proof. □

**Proof of Lemma A.2.** The claim on $\hat A_\gamma/\hat A$ follows from Taylor expansion and the explicit, $L$-independent formula (A.12) (see [14, Lemma 2.5]). The claim on $\hat F_\gamma/\hat F$ follows from Lemma 3.3, with $F = \delta - D$ and $\beta = 0$. We cannot immediately apply our previous bounds to $\hat E_\gamma/(\hat A \hat F)$, because now it mixes both $P$ and $D$, and in particular we now have the two different infrared bounds

$$\hat A(k) \gtrsim |k|^2, \qquad \hat F(k) \gtrsim L^2|k|^2 \wedge 1. \tag{A.16}$$

We adapt the proof of Lemma 3.3 to bound $\hat{E}_\gamma/(\hat{A}\hat{F})$, as follows. By the infrared bounds,

$$\left|\frac{\hat{E}_\gamma}{\hat{A}\hat{F}}(k)\right| \lesssim L^{-2}|k|^{-4}\big|\hat{E}_\gamma(k)\big|\mathbb{1}_{B_L} + |k|^{-2}\big|\hat{E}_\gamma(k)\big|. \tag{A.17}$$

Recall that $E$ is given by

$$E = A - \sigma^{-2}F = (\delta - P) - \sigma^{-2}(\delta - D), \tag{A.18}$$

so we have $\sum_{x\in\mathbb{Z}^d}|x|^2|E(x)| \le 1+1=2$. Also, by the norm estimates (3.21) of $\hat{D}_\gamma$, we have

$$\|\hat{E}_\gamma\|_r \lesssim 1 + L^{-2}\big(1+L^{|\gamma|-d/r}\big) \le 2 + L^{|\gamma|-2-d/r} \quad \big(0\le r^{-1}<1\big). \tag{A.19}$$

Let $\tau\in(0,2)$ and $q=q_2\in(\frac{2-\tau+|\gamma|}{d},1)$. We start with the second term of (A.17). Suppose first that $|\gamma|<\tau$, so $|\gamma|\in\{0,1\}$. By symmetry and $\sum_{x\in\mathbb{Z}^d}E(x)=0$, it follows that $\hat{E}(k)=\sum_{x\in\mathbb{Z}^d}E(x)(\cos(k\cdot x)-1)$. Taylor expansion of $\cos(k\cdot x)-1$ or its derivative in $k$ gives

$$\big|\hat{E}_\gamma(k)\big| \lesssim \sum_{x\in\mathbb{Z}^d}|k|^{2-|\gamma|}|x|^2\big|E(x)\big| \lesssim |k|^{2-|\gamma|}, \tag{A.20}$$

so that the $L^q$ norm of $|k|^{-2}|\hat{E}_\gamma(k)|\lesssim|k|^{-|\gamma|}$ can be bounded independent of $L$ when $q^{-1}>|\gamma|/d$. This includes the desired $q$'s because $\tau\le 2$. If $|\gamma|\ge\tau$, we observe that every $q$ with $\frac{2-\tau+|\gamma|}{d}<q^{-1}<1$ can be written in the form $q^{-1}=r^{-1}+p^{-1}$ for some $r^{-1}\in(\frac{|\gamma|-\tau}{d},1)$ and $p^{-1}>2/d$. Then (A.19) and Hölder's inequality gives

$$\big\||k|^{-2}\hat{E}_\gamma(k)\big\|_q \le \big\||k|^{-2}\big\|_p\|\hat{E}_\gamma\|_r \lesssim 1+L^{|\gamma|-2-d/r} \le 1+L^{\tau-2}\le 2. \tag{A.21}$$

Therefore, the $L^q$ norm of the second term on the right-hand side of (A.17) is bounded uniformly in $L$.

For the first term of (A.17), suppose first that $|\gamma|<2+\tau$. By (A.18) and the decay (3.19) of $D$,

$$\big|E(x)\big| \lesssim \frac{1+L^{-2}(1+L^4)}{|\!|\!|x|\!|\!|^{d+4}} \lesssim \frac{L^2}{|\!|\!|x|\!|\!|^{d+4}}. \tag{A.22}$$

Lemma 3.4 with $\rho=2$ then gives $|\hat{E}_\gamma(k)|\lesssim L^2|k|^{2+\tau-|\gamma|}$. Since the $L^2$ cancels with $L^{-2}$, the $L^q(B_L)$ norm of the first term on on the right-hand side of (A.17) is exactly that of $|k|^{2+\tau-|\gamma|-4}=|k|^{-(2-\tau+|\gamma|)}$, which is of order $L^{2-\tau+|\gamma|-d/q}\le 1$ for $q^{-1}>(2-\tau+|\gamma|)/d$, as desired. For the remaining case $|\gamma|\ge 2+\tau$, Hölder's inequality and (A.19) imply that the $L^q$ norm of the first term on the right-hand side of (A.17) is bounded by

$$\frac{1}{L^2}\|\hat{E}_\gamma\|_r\big\||k|^{-4}\mathbb{1}_{B_L}\big\|_p \lesssim \frac{1}{L^2}\big(1+L^{|\gamma|-2-d/r}\big)L^{4-d/p} \lesssim L^{|\gamma|-d/q} \tag{A.23}$$

for $q^{-1}=r^{-1}+p^{-1}$, $r\in(1,\infty]$, $\frac{|\gamma|-2-\tau}{d}<r^{-1}\le\frac{|\gamma|-2}{d}$, and $p^{-1}>4/d$. In particular, the bound holds for all $q^{-1}>(2-\tau+|\gamma|)/d$. The desired bound then follows from $\tau\le 2$. This completes the proof. □

## Appendix B: Proof of Lemma 3.6

Lemma 3.6 makes the following assertions for $D$ defined by Definition 1.1, for $L\ge L_0$ with $L_0$ sufficiently large depending only on $d$ and $v$:

$$D(x) \lesssim \frac{L^a}{|\!|\!|x|\!|\!|^{d+a}} \quad \text{for any } a>0, \tag{B.1}$$

$$\hat{A}_\mu(k)-\hat{A}_\mu(0) \gtrsim L^2|k|^2\wedge 1 \quad \text{for all } k\in\mathbb{T}^d, \text{ uniformly in } \mu\in\left[\frac{1}{2},1\right], \tag{B.2}$$

$$\|\hat{D}_\alpha\|_q \lesssim L^{|\alpha|-d/q} \quad \text{for each } \alpha \text{ and for all } 0\le q^{-1}<1. \tag{B.3}$$

**Proof.** For (B.1), let $b > 0$. By definition, $D(x) \lesssim L^{-d} \mathbb{1}_{\|x\|_\infty \le L}$, so $D(x) \lesssim L^{-d}(L/|x|)^b$, and the desired result follows by choosing $b = d + a$.

The infrared bound (B.2) is proved in [10, Appendix A].

It remains to prove (B.3). For $q = \infty$, we simply observe that $|\hat{D}_\alpha(k)| \le \sum_{x \in \mathbb{Z}^d} |x^\alpha| D(x) \le L^{|\alpha|}$. For $q < \infty$, we divide the integral according to whether or not $\|k\|_\infty \le 1/L$. When $\|k\|_\infty \le 1/L$, we use $|\hat{D}_\alpha(k)| \le L^{|\alpha|}$. Since the volume is of order $L^{-d}$, we obtain the desired upper bound $L^{|\alpha|-d/q}$. When $\|k\|_\infty > 1/L$, we apply [6, (5.34)] (whose proof generalises to any number of derivatives) as in the proof of [6, Lemma 5.7], as follows. The domain $\|k\|_\infty > 1/L$ is the disjoint union over nonempty subsets $S \subset \{1, \dots, d\}$ of

$$R_S = \{k \in \mathbb{R}^d : 1/L < k_i \le \pi \text{ for } i \in S, |k_j| \le 1/L \text{ for } j \notin S\}. \tag{B.4}$$

By [6, (5.34)], for $q \in (1, \infty)$,

$$\begin{aligned} \int_{R_S} \big|\hat{D}_\alpha(k)\big|^q \, dk &\lesssim L^{q|\alpha|} \int_{R_S} \prod_{i \in S} |Lk_i|^{-q} \, dk \\ &\lesssim L^{q(|\alpha|-|S|)} \left( \int_{1/L}^{\pi} t^{-q} \, dt \right)^{|S|} \left( \int_0^{1/L} 1 \, dt \right)^{d-|S|} \\ &\lesssim L^{q(|\alpha|-|S|)} L^{(q-1)|S|} L^{|S|-d} = L^{q|\alpha|-d}. \end{aligned} \tag{B.5}$$

The desired result (B.3) then follows by summing over $S$. □

## Funding

The work of both authors was supported in part by NSERC of Canada.

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