Exercise for Talk 1: Introduction to modular curves and Siegel modular varieties

In this talk, we will discuss basics of moduli space of abelian varieties with additional structures: PEL for polarizations, endomorphisms, and level structure. We will start with the case of modular curve, then the Siegel space, then move on to the case of unitary Shimura varieties, and finally Hilbert modular varieties.

Problem 1.1 (Siegel half space versus Hodge filtration)**.** Complete the proof of description of $\mathbb{C}$-points of Siegel space. In particular, explain the following two points:

(1) Why is providing a Hodge filtration for abelian varieties equivalent to giving a complex structure on $\Lambda \otimes \mathbb{R}$?
(2) Deduce that, if $J = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ gives the complex structure on $\Lambda \otimes \mathbb{R}$, then $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)(iI_g)$ belongs to $\mathfrak{H}_g^{\pm}$.

Problem 1.2 (Siegel space as homogeneous)**.** (1) Consider $\mathrm{Sp}_{2g}(\mathbb{R})$ acting on $\mathfrak{H}_g$ given by $Z \mapsto (AZ+B)(CZ+D)^{-1}$. Show that this action is well-defined. What is the centralizer?

(2) Similarly consider the $\mathrm{GSp}_{2g}(\mathbb{R})$-action on $\mathfrak{H}_g^{\pm}$. What is the centralizer? Observe that this action factors through $\mathrm{PSp}_{2g}(\mathbb{R})$. Give an example of an element which turns $\mathfrak{H}_g$ into $\mathfrak{H}_g^{-}$.

What we are getting at here is a small subtlety for Shimura varieties, which we will encounter later. Let G be a reductive group over $\mathbb{R}$; the locally Hermitian space X we consider is technically $G_{\mathrm{ad}}(\mathbb{R})/K_{\mathrm{ad}}$, where G_{ad} is the quotient of G by its center, and K_{ad} is the maximal compact subgroup of G_{ad}. So $G(\mathbb{R})$ naturally acts on X and the action factors through $G_{\mathrm{ad}}(\mathbb{R})$. But the image $G(\mathbb{R})$ in $G_{\mathrm{ad}}(\mathbb{R})$ is typically only a connected component. (3) Explain the case when $G = \mathrm{Sp}_{2g,\mathbb{R}}$ using the exact sequence $1 \to Z(G) \to G \to G_{\mathrm{ad}} \to 1$.

Problem 1.3 (quasi-isogeny versus lattices)**.** (1) Fix an elliptic curve E_0 over $\mathbb{C}$. Show that there is an equivalence of categories:

$$\begin{array}{ccc} \{\text{Elliptic curves } E \text{ with a quasi-isogeny } \alpha : E \to E_0\} & \xrightarrow{\cong} & \{\widehat{\mathbb{Z}}\text{-lattices in } \widehat{V}(E_0) := \widehat{T}(E_0) \otimes_{\mathbb{Z}} \mathbb{Q}\} \\ E & \longmapsto & \widehat{T}(E) \end{array}$$

(2) Suppose now that E_0 is an elliptic curve over $\mathbb{Q}$. Show that, under the correspondence above, a $\widehat{\mathbb{Z}}$-lattice Λ of $\widehat{V}(E_0)$ is stable under the $\mathrm{Gal}_{\mathbb{Q}}$-action if and only if it comes from an elliptic curve over $\mathbb{Q}$.

(3) Let A_0 be an abelian variety over $\mathbb{C}$ with principal polarization $\lambda_0 : A_0 \xrightarrow{\cong} A_0^{\vee}$. Show that there is an equivalence of categories:

$$\begin{array}{ccc} \left\{\begin{matrix}\text{Abelian varieties } A \text{ with a quasi-isogeny } \alpha : A \to A_0 \\ \text{together with a principal polarization } \lambda : A \to A^{\vee} \\ \text{such that } \lambda = \alpha^{\vee} \circ \lambda_0 \circ \alpha\end{matrix}\right\} & \xrightarrow{\cong} & \left\{\begin{matrix}\widehat{\mathbb{Z}}\text{-lattices } \Lambda \text{ in } \widehat{V}(A_0) \text{ which is} \\ \text{self-dual under the symplectic pairing}\end{matrix}\right\} \\ A & \longmapsto & \widehat{T}(A) \end{array}$$

Problem 1.4 (Rationalized moduli problem)**.** This problem discusses a somewhat subtle difference in writing for two moduli problems, but it leads to interesting simplifications.

Recall that the scheme $Y_1(N)$ represents the functor

$$\begin{aligned}\mathcal{M} : \mathbf{Sch}_{/\mathbb{Q}}^{\text{loc. noe.}} &\longrightarrow \mathbf{Sets}\\ S \longmapsto \mathcal{M}(S) &= \left\{\begin{array}{l}\text{isom. classes of } (E,\eta); \quad E \text{ is an elliptic curve over } S;\\ \text{choosing a geom. point } \bar{s} \text{ on each conn. component of } S\\ \eta : \widehat{\mathbb{Z}}^{\oplus 2} \xrightarrow{\simeq} \widehat{T}(E) \text{ is a } \pi_1(S,\bar{s})\text{-stable } \widehat{\Gamma}_1(N)\text{-orbit of isoms.}\end{array}\right\}.\end{aligned}$$

We need a variant of this. Show that $Y_1(N)$ also represents the following functor

$$\begin{aligned}\mathcal{M}' : \mathbf{Sch}_{/\mathbb{Q}}^{\text{loc. noe.}} &\longrightarrow \mathbf{Sets}\\ S \longmapsto \mathcal{M}(S) &= \left\{\begin{array}{l}\textit{equiv.} \text{ classes of } (E',\eta'); \quad E' \text{ is an elliptic curve over } S;\\ \text{choosing a geom. point } \bar{s} \text{ on each conn. component of } S\\ \eta' : \mathbb{A}_f^{\oplus 2} \xrightarrow{\simeq} \widehat{V}(E') \text{ is a } \pi_1(S,\bar{s})\text{-stable } \widehat{\Gamma}_1(N)\text{-orbit of isoms.}\end{array}\right\}.\end{aligned}$$

Here $\widehat{V}(E') := \widehat{T}(E') \otimes_{\mathbb{Z}} \mathbb{Q}$ is the rational version of the Tate module. Two pairs (E',η') and (E'',η'') are equivalent if there is a quasi-isogeny $\alpha : E' \dashrightarrow E''$ such that $\alpha \circ \eta' = \eta''$ (as K-orbits). In other words, we are classifying elliptic curves up to quasi-isogenies.

The benefit of the second moduli problem is that it makes sense for any open compact subgroup K of $\mathrm{GL}_2(\mathbb{A}_f)$ in place of $\widehat{\Gamma}_1(N)$, without the constraint that $K \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$.[1]

Additional question: how to formulate this new type of moduli problems over $\mathbb{Z}_{(p)}$ when we need K to be of the form $K = K^p\,\mathrm{GL}_2(\mathbb{Z}_p)$ for K^p an arbitrary open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$?

What about the Siegel case?

Problem 1.5 (Γ_0-level structure)**.** We give another moduli interpretation of modular curve with $\Gamma_0(p)$-level structure, when p is a prime number.

(1) Show that the following two functors are equivalent.

$$\begin{aligned}\mathcal{M}, \mathcal{M}' : \ \mathbf{Sch}_{/\mathbb{Z}_{(p)}} &\longrightarrow \mathbf{Sets}\\ S \longmapsto \mathcal{M}(S) &= \left\{\begin{array}{l}\text{isomorphism classes of isogenies } \beta : E \to E'\\ \text{of degree } p \text{ between two elliptic curves over } S\end{array}\right\}.\\ S \longmapsto \mathcal{M}'(S) &= \left\{\begin{array}{l}\text{isomorphism classes of } (E,C):\\ E \text{ is an elliptic curve over } S\\ C \text{ is a subgroup of } E[p] \text{ of degree } p\end{array}\right\}.\end{aligned}$$

[1]This is not a serious issue for GL_2-theory, as there is a nice well-behaved oldform / newform theory, namely, for every holomorphic automorphic representation, we can find a subspace of newform "representing this automorphic representation. Beyond GL_2, there seems to be no such nice theory of newform/oldform. Therefore, it is important to allow more general level structures.

They are represented by a stack[2] $Y_0(p)$ over $\mathbb{Z}_{(p)}$ (but not smooth over the fiber at p). This will not give a scheme, as we will see in Problem 1.6; however we can "pretend" that it is a scheme for most purpose. We will come to study its geometry later.

(2) Using either moduli problem, explain what the Hecke correspondence at p looks like.

(3) How to generalize moduli problem $\mathcal{M}$ to Siegel moduli varieties? (Note that the polarization will play a role, and also there are many parabolic subgroups of $\mathrm{GSp}_{2g}(\mathbb{F}_p)$.)

Remark: Unfortunately, this type of moduli problem does not generalize so easily to, say $\Gamma_0(p^2)$-level structure. The problem is: for a subgroup C of degree p^2, there is no way to exclude the possibility that $C = E[p]$. Note that $C = E[p]$ if and only if C is p-torsion, which is a closed condition. But the moduli problem might have non-reduced special fiber. One cannot "remove" a closed subscheme of a non-reduced scheme. (One can still make a definition, but it wouldn't be the one that you want.)

Problem 1.6 (Quadratic twists of elliptic curves)**.** We discuss the question of quadratic twist of elliptic curves.

Classical definition: For an elliptic curve $E : y^2 = x^3 + ax + b$ over $\mathbb{Q}$, a *quadratic twist* is the elliptic curve $E_D : Dy^2 = x^3 + ax + b$ for some $D \in \mathbb{Q}$ typically square-free. The two curves E and E_D are not isomorphic over $\mathbb{Q}$ but are isomorphic over $\mathbb{Q}(\sqrt{D})$. A key feature is that there is a j-invariant attached to D as follows: the modular function $j : \mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H} \xrightarrow{\cong} \mathbb{C}$ gives a bijection. (Here I used double slash to indicate "coarse moduli problem"; we may temporarily ignore this now.) The statement above amounts to say $j(E) = j(E_D)$.

Moreover, via the isomorphism $j : \mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H} \xrightarrow{\cong} \mathbb{C}$, we can endow $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H}$ a natural $\mathbb{Q}$-structure (namely, a rational point on it means a point with j-invariant in $\mathbb{Q}$.) But we still write $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H}$ for it to mean the corresponding $\mathbb{Q}$-scheme.

Galois cohmology explanation: Elliptic over $\mathbb{C}$ (or over $\overline{\mathbb{Q}}$) up to isomorphism are determined by the j-invariant.

(1) Prove that, given an elliptic curve E over $\mathbb{Q}$, any other elliptic curves that are isomorphic to E over $\overline{\mathbb{Q}}$ but not over $\mathbb{Q}$, called *forms of* E, are classified by $H^1(\mathbb{Q}, \mathrm{Aut}_{\overline{\mathbb{Q}}}(E_{\overline{\mathbb{Q}}}))$.

(2) Find $\mathrm{Aut}_{\overline{\mathbb{Q}}}(E_{\overline{\mathbb{Q}}})$ for all $E_{\overline{\mathbb{Q}}}$. Show that unless $j(E) = 0$ or 1728, $\mathrm{Aut}_{\overline{\mathbb{Q}}}(E_{\overline{\mathbb{Q}}}) = \{\pm 1\}$. Deduce that in this case, all forms of E are quadratic twists.

Explanation using moduli stack: (Let use try if this explanation makes sense.) If we consider the moduli problem of elliptic curves, call it $\mathcal{M}$, it is represented by a stack. On an open subset, it looks like $U/\{\pm I_2\}$ where U is an open subset of $\mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H} - \mathrm{SL}_2(\mathbb{Z})\{i, e^{2\pi i/3}\}$. Here $\pm I_2$ acts trivially on U. But as a stack, it is natural to keep this quotient. In other words, we have a natural morphism $\mathcal{M} \to \mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H}$. Again, this can be defined over $\mathbb{Q}$.

Giving a j-invariant (say over $\mathbb{Q}$ but not at 0 or 1728) amounts to a morphism $x : \operatorname{Spec}\mathbb{Q} \to \mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H}$, we can take the fiber product:

$$\begin{array}{ccc} \big[\operatorname{Spec}\mathbb{Q}/\{\pm 1\}\big] & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{Q} & \xrightarrow{\ x\ } & \mathrm{SL}_2(\mathbb{Z})\backslash\backslash\mathfrak{H}. \end{array}$$

[2]It is a stack but not a scheme because the moduli problem is supposed to be the quotient $Y_0(p)/\{1, \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)\}$, so every point has nontrivial automorphism. Or in the language of moduli problem, $[-1] : E \to E$ is an automorphism of a pair (E, C) but it induces trivial map on $\mathcal{M}$ if this were represented by a scheme.

Again, $\big[\operatorname{Spec}\mathbb{Q}/\{\pm1\}\big]$ is the stack given by "quotienting" $\operatorname{Spec}\mathbb{Q}$ by the trivial $\{\pm1\}$-action. In the fancier language, this is the classifying space for $\{\pm1\}$. So a $\operatorname{Spec}\mathbb{Q}$-point of $\big[\operatorname{Spec}\mathbb{Q}/\{\pm1\}\big]$ corresponds to a $\{\pm1\}$-torsor over $\operatorname{Spec}\mathbb{Q}$, that is a quadratic extension of $\mathbb{Q}$ (including $\mathbb{Q}\times\mathbb{Q}$).

Explicitly, for a quadratic extension $\mathbb{Q}(\sqrt{D})$, we have $\iota_D : \operatorname{Spec}\mathbb{Q}(\sqrt{D}) \to \operatorname{Spec}\mathbb{Q}$, equivariant for the $\{\pm1\}$-action, where -1 acts by natural Galois action on $\mathbb{Q}(\sqrt{D})$ and trivially on $\mathbb{Q}$. Taking the quotient of ι_D by the $\{\pm1\}$-action gives $\iota_D : \operatorname{Spec}\mathbb{Q} \to \big[\operatorname{Spec}\mathbb{Q}/\{\pm1\}\big]$.

Problem 1.7 (Hodge structure)**.** A *Hodge structure* on a finite dimensional $\mathbb{R}$-vector space V is a bi-grading decomposition of $\mathbb{C}$-vector spaces

$$V \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{p,q\in\mathbb{Z}} V^{pq} \tag{1.7.1}$$

such that $\overline{V^{pq}} = V^{qp}$. (Here the complex conjugation is with respect to the complex structure on $V\otimes\mathbb{C}$,namely $V\otimes_{\mathbb{R}}\mathbb{C} \to V\otimes_{\mathbb{R}}\mathbb{C}$ sending $z\otimes r \mapsto z\otimes\bar{r}$.)

(1) Show that the Hodge structure on V is equivalent to an algebraic homomorphism

$$h : \mathbb{S} := \operatorname{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to \operatorname{GL}_{\mathbb{R}}(V) \tag{1.7.2}$$

where $\operatorname{Res}_{\mathbb{C}/\mathbb{R}}$ is the Weil restriction, (i.e. a homomorphism $h_{\mathbb{C}} : (\operatorname{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}})_{\mathbb{C}} \cong \mathbb{C}^\times\times\mathbb{C}^\times \to \operatorname{GL}(V_{\mathbb{C}})$ that commutes with complex conjugation.) The action of $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ on V^{pq} is given by $h(z)v = z^{-p}\bar{z}^{-q}$. [3]

(2) Consider the homomorphism (which is the Siegel case)

$$\begin{aligned} h : \mathbb{C}^\times &\longrightarrow \operatorname{GSp}_{2g}(\mathbb{R}) \\ z = x+iy &\longmapsto \begin{pmatrix} xI_g & yI_g \\ -yI_g & xI_g \end{pmatrix}, \end{aligned}$$

where the symplectic space $V = \mathbb{R}^{2g}$ uses the symplectic form $\left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right)$. Under the correspondence in (1), what is the corresponding Hodge decomposition (1.7.1) in terms of coordinates?

The *Hodge types* of a Hodge structure are those (p,q) where $V^{pq} \neq 0$, counted with multiplicity $\dim V^{pq}$. We say that V has *pure weight* n if for all of its Hodge type (p,q), we have $p+q=n$.

(3) Given a Hodge structure on V of pure weight n defines a *decreasing* Hodge filtration on $V_{\mathbb{C}}$ by

$$F^pV_{\mathbb{C}} := \bigoplus_{r\geq p} V^{rs}.$$

Show that in this case, one may recover the bigrading decomposition by $V^{pq} = F^pV \cap \overline{F^qV}$.

For a subring $A\subseteq\mathbb{R}$, a *A-Hodge structure* is a finite type A-module V with a Hodge structure on $V\otimes_A\mathbb{R}$.

We write $\mathbb{Z}(1)$ for $(2\pi i)\mathbb{Z}\subset\mathbb{C}$, we provide it with Hodge type $(-1,-1)$, so weight -2, and call it the *Tate Hodge module.* For $n\geq1$, set $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$ and $\mathbb{Z}(-n) = \mathbb{Z}(n)^\vee$. A polarization of a $\mathbb{Z}$-Hodge structure of pure weight n is a bilinear pairing

$$\psi : V\times V \to \mathbb{Z}(-n)$$

[3] As we discussed earlier, this point of view is very important later. A Shimura datum will be consisting of a reductive group G over $\mathbb{Q}$ and a conjugacy classes of homomorphisms $h:\mathbb{S}\to G_{\mathbb{R}}$.

such that $(x, y) \mapsto (2\pi i)^n \psi_{\mathbb{R}}(x, h(i)y)$ is a symmetric positive definite pairing, in the sense that $(2\pi i)^n \psi_{\mathbb{R}}(x, h(i)x) \in (2\pi i)^n \mathbb{R}_{>0}$ for every $x \in V_{\mathbb{R}} \backslash \{0\}$. (Here h is the map (1.7.2).)

(4) Show that ψ itself is symmetric if n is even and alternating if n is odd.

Remark: the standard convention is: for X a smooth variety over $\mathbb{C}$, $H^n(X, \mathbb{C}) = \oplus_{p+q=n} H^{pq}$, each H^{pq} has Hodge type (p, q). By duality, $H_n(X, \mathbb{C})$ has type $(-p, -q)$ with $p+q = n$. The Tate Hodge module can be viewed as $H_1(\mathbb{G}_m)$ through the exponential map $\mathbb{C}/\mathbb{Z}(1) \to \mathbb{C}^\times$.

(5) Show that there is a one-to-one correspondence between abelian varieties with a polarization and $\mathbb{Z}$-Hodge structures with polarization of type $\{(-1, 0), (0, -1)\}$.

(6) Following Problem 1.2 and part (2) above, identify $\mathfrak{H}_g^{\pm}$ with the $\mathrm{PSp}_{2g}(\mathbb{R})$-orbit of homomorphism $\mathbb{S} \to \mathrm{GSp}_{2g,\mathbb{R}}$, where $\mathrm{PSp}_{2g}(\mathbb{R})$ acts on this set by acting on $\mathrm{GSp}_{2g,\mathbb{R}}$ via conjugation.

Problem 1.8 (Fine moduli problem)**.** While writing the moduli problem for elliptic curves, we often assume that the level is $\Gamma_1(N)$ for $N \geq 4$. This is because for Γ sufficiently small (e.g. $\Gamma = \Gamma(N)$ for $N \geq 4$), the moduli problem is representable, say by $\mathcal{M}_\Gamma$, but when we consider moduli stack for $\Gamma_1(N)$-level structure, we essentially need to take a quotient $\mathcal{M}_\Gamma/(\Gamma_1(N)/\Gamma)$. As a quasi-projective variety, this quotient is well-defined (as $\mathrm{Spec}(\mathcal{O}_{\mathcal{M}_\Gamma})^{\Gamma_1(N)/\Gamma}$), but the action of $\Gamma_1(N)/\Gamma$ may not act freely on points of $\mathcal{M}_\Gamma$.

In practice, we need to look at stabilizers of $\tau = i$ and $\tau = e^{2\pi i/3}$. Show that their stabilizer groups in $\mathrm{GL}_2(\mathbb{Z})$ have order 4 and 6, respectively, and their intersection with $\Gamma_1(N)$ is trivial if $N \geq 3$. In general, show that the stabilizer group of any $\tau \in \mathcal{H}$ in $\mathrm{GL}_2(\mathbb{Z})$ has trivial intersection with $\Gamma_1(N)$ if $N \geq 4$.

This is why we need $N \geq 3$ to ensure that the moduli problem is representable by a scheme (as opposed to a stack). Without this condition, we always get a Deligne–Mumford stack, a rather nice one, namely a global quotient of a scheme by a finite group.

Hints.

Problem 1.2: See for example, Lan, Kai-wen's paper Example-based introduction to Shimura varieties, §3.1.1.

For (3), when $G = \mathrm{Sp}_{2g}(\mathbb{R})$, $Z(G) = \{\pm 1\}$, taking the Galois cohomology for $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of the exact sequence $1 \to \{\pm 1\} \to G \to G_{\mathrm{ad}} \to 1$, we get

$$\mathrm{Sp}_{2g}(\mathbb{R}) \to \mathrm{PSp}_{2g}(\mathbb{R}) \to H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \{\pm 1\}) \to \cdots$$

It is easy to see that $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \{\pm 1\}) = \{\pm 1\}$; this is related to the fact that the image of $\mathrm{Sp}_{2g}(\mathbb{R})$ in $\mathrm{PSp}_{2g}(\mathbb{R})$ is only one connected component, and the quotient is $\{\pm 1\}$.

Problem 1.4: Imitate the proof of $Y_1(N)(\mathbb{C}) \cong \mathrm{GL}_2(\mathbb{Q})\backslash \mathfrak{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_f)/\widehat{\Gamma}_1(N)$, using moduli problems, Especially, the argument to construct the reverse map. We here give another point of view of the construction therein, regarding choosing isogeny to match the lattice.

Another way to define Tate module is $\widehat{T}(A) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, A)$. So when we have an isogeny $\alpha : A \to A'$ with finite kernel G, we may apply Hom to the exact sequence $0 \to G \to A \to A' \to 0$ to get

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, G) & \longrightarrow & \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, A) & \longrightarrow & \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, A') & \longrightarrow & \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G) \\ & \| & & \| & & \| & & \| \\ & 0 & & \widehat{T}(A) & & \widehat{T}(A') & & G, \end{array}$$

where the last isomorphism comes form applying $\mathrm{Hom}(-, G)$ to the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ to deduce that $\mathrm{Hom}(\mathbb{Z}, G) \cong \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$.

Now go back to the discussion from the lecture, where we consider $0 \to G \to E_\tau \to E \to 0$, with G corresponding to $\frac{M_1 \widehat{\mathbb{Z}}^{\oplus 2}}{M_1 M_2 \widehat{T}(E_\tau)}$. Then E is the elliptic curve that fits canonically in the following exact sequence

$$0 \to \widehat{T}(E_\tau) \to \widehat{T}(E) \to \frac{M_1 \widehat{\mathbb{Z}}^{\oplus 2}}{M_1 M_2 \widehat{T}(E_\tau)}$$

(For degree reasons, this map must be right exact. It then follows that the natural map $\widehat{T}(E_\tau) \to \widehat{T}(E)$ can be naturally identified with $M_2 \widehat{T}(E_\tau) \to \widehat{\mathbb{Z}}^{\oplus 2}$.) We can reverse the argument to see why we chose the subgroup G as in the construction.

For the "rational" moduli problem over $\mathbb{Z}_{(p)}$ for level $K = K^p \, \mathrm{GL}_2(\mathbb{Z}_p)$, we use the following:

$$\mathcal{M}' : \mathbf{Sch}^{\text{loc. noe.}}_{/\mathbb{Z}_{(p)}} \longrightarrow \mathbf{Sets}$$

$$S \longmapsto \mathcal{M}(S) = \left\{ \begin{array}{l} \textit{equiv.}\text{ classes of } (E', \eta'); \quad E' \text{ is an elliptic curve over } S; \\ \text{choosing a geom. point } \bar{s} \text{ on each conn. component of } S \\ \eta' : \mathbb{A}_f^{(p)\oplus 2} \xrightarrow{\simeq} \widehat{V}^{(p)}(E') \text{ is a } \pi_1(S, \bar{s})\text{-stable } K^p\text{-orbit of isoms.} \end{array} \right\}.$$

Here $V^{(p)}(E')$ is the rationalized Tate module away from p, and we say that (E', η') and (E'', η'') are equivalent if there is a prime-to-p quasi-isogeny $\alpha : E' \dashrightarrow E''$ such that $\eta'' = \alpha \circ \eta'$ (as K^p-orbits).

Here is somewhat the principle: it is very bad to talk about p-adic Tate modules in characteristic p, because for an elliptic curve over $\mathbb{F}_p$, Tate module picks up $E[p](\overline{\mathbb{F}}_p)$, which is certainly not $(\mathbb{Z}/p\mathbb{Z})^{\oplus 2}$ as you "want" it to be, even worse, the isomorphism classes of $E[p]$ depends on whether E is ordinary or supersingular. Writing a condition on the isomorphism classes of $E[p]$ is likely to give a "subspace of what you want the moduli space to be". (Namely, the moduli problem may still be representable, but it may not proper over the base even if you compactify at the cusps.

For the Siegel case, one needs to note that the polarization is no longer required to be a genuine isogeny, but only a quasi-isogeny, that is a map like $A \xleftarrow{\times m} A \to B$.

Problem 1.5: (2) Just send the pair (E, E') to E and to E', respectively.

(3) To generalize this, as usual, we need to add subgroups $C \subset A[p]$ or more generally filtrations $C_1 \subset C_1 \subset \cdots \subset A[p]$ into the moduli problem. More importantly, we also need to require these subgroups C_i to be *isotropic* with respect to the polarization λ.

Caveat: In the Siegel case, if one wants to use moduli problem to explain Hecke operators, one has to be very careful, because when we change A to A/C, say, the polarization will no longer be principal. This is less of a problem if we use the moduli problem of the form in Problem 1.4, but still, if we want to write this over $\mathbb{Z}_{(p)}$ while making a p-isogeny, this is rather subtle.

Problem 1.7: The main reference for Hodge structure is §1.1 of

P. Deligne, Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques, in *Automorphic forms, representations and L-functions*, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 247–289.

One can also read Chap 6 of Milne's Introduction to Shimura varieties.

(6) The map h defined in (2) corresponds to $iI_g \in \mathfrak{H}_g$.

Problem 1.8: If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ fixes $z \in \mathfrak{H}$, it means that $\frac{az+b}{cz+d} = z$. it then follows that $cz^2 + (d-a)z - b = 0$. But we need an imaginary solution, so

$$\Delta = (d-a)^2 + 4bc = (d+a)^2 - 4 < 0$$

It then follows that $|a+d| < 2$. But if $c \equiv 1, d \equiv 0 \pmod N$, we can deduce that $a \equiv d \equiv 1 \pmod N$. Then $a + d \equiv 2 \pmod N$. When $N \geq 4$, this contradicts the discussion above. When $N = 3$, the matrix $\left(\begin{smallmatrix} -2 & -1 \\ 3 & 1 \end{smallmatrix}\right) \in \Gamma_1(3)$ (which has order 3) has a fixed point on $\mathfrak{H}$.