Exercise for Talk 2: Unitary and Hilbert modular varieties and geometric Hilbert modular forms

Continued with the previous talk, we introduce the definition of unitary Shimura varieties and Hilbert modular varieties. Then we move to explain modular forms in terms of sections of automorphic line bundles, and discuss Katz' geometric interpretation. After this, we generalize this to the case of Hilbert modular varieties and possibly general Shimura varieties.

Problem 2.1 (More general unitary Shimura varieties). Let E be a CM field, that is a totally imaginary extension of a totally real field F. Let c denote the complex conjugation on $\mathbb{C}$; it induces the complex conjugation on E (through any embedding of E into $\mathbb{C}$).

Let V be a Hermitian space of dimension n over E/F. There is a natural unitary group $G_0 := U_F(V)$ defined over F. But we will consider the similitude unitary group $G = \mathrm{GU}(V)$ *over* $\mathbb{Q}$ *with similitudes in* $\mathbb{Q}$: for a $\mathbb{Q}$-algebra R, define

$$G(R) = \big\{(g, c) \in \mathrm{GL}(V \otimes_{\mathbb{Q}} R) \times R^\times \;\big|\; \langle gx, gy\rangle = c\langle x, y\rangle \text{ for } x, y \in V \otimes_{\mathbb{Q}} R\big\}.$$

(1) Prove that we have an exact sequence

$$1 \to \mathrm{Res}_{F/\mathbb{Q}} G_0 \to \mathrm{GU}(V) \to \mathbb{G}_m \to 1.$$

Just to emphasize again, G_0 is a group scheme over F, yet $\mathrm{GU}(V)$ is a group scheme over $\mathbb{Q}$.

A *CM type* of E is a subset $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ such that $\mathrm{Hom}(E, \mathbb{C}) = \Phi \coprod c(\Phi)$, or equivalently, for every embedding $\tau : E \to \mathbb{C}$, exactly one of τ and $c \circ \tau \in \Phi$. Fix a CM type Φ.

(2) Convince yourself that there exists an element $\delta \in E^{c=-1}$ such that for each embedding $\tau \in \Phi$, $\tau(\delta) \in \mathbb{R}_{>0} i$ (so that for those $\tau \notin \Phi$, $\tau(\delta) \in \mathbb{R}_{<0} i$).

Using this element δ, we define an alternating E-Hermitian and $\mathbb{Q}$-valued form:

$$\begin{aligned} \{-,-\} :\ & V \times V \longrightarrow \mathbb{Q} \\ \{x, y\} &= \mathrm{Tr}_{E/\mathbb{Q}}(\delta\langle x, y\rangle). \end{aligned}$$

(3) First show that $\{-,-\}$ is an alternating bilinear form and $\{\alpha x, y\} = \{x, c(\alpha) y\}$. Then prove that the association $\langle -,-\rangle \mapsto \{-,-\}$ gives a one-to-one correspondence between Hermitian forms on V and alternating E-Hermitian and $\mathbb{Q}$-valued forms on V. (This bijection holds for any given element $\delta \in E^\times$.)

We consider the isomorphism

$$V_{\mathbb{R}} \cong \prod_{\tau\in\Phi} V \otimes_{E,\tau} \mathbb{C} \simeq \prod_{\tau\in\Phi} \mathbb{C}^n,$$

which identifies $V_{\mathbb{R}}$ with $[F : \mathbb{Q}]$ copies of $\mathbb{C}^n$; but this depends crucially on the choice of CM type Φ. We assume that the latter isomorphism is taken so that the Hermitian form on the τ-factor $\mathbb{C}^n$ is given by $\left(\begin{smallmatrix} I_{a_\tau} & \\ & -I_{b_\tau}\end{smallmatrix}\right)$, where $n = a_\tau + b_\tau$.

Consider a homomorphism

$$\begin{aligned} h_0 :\ \mathbb{C}^\times &\longrightarrow \mathrm{GU}(V)(\mathbb{R}) \subset \mathrm{GL}(V_{\mathbb{R}}) \cong \textstyle\prod_{\tau\in\Phi} \mathrm{GL}_n(\mathbb{C}) \\ z &\longmapsto \begin{pmatrix} z \cdot I_{a_\tau} & \\ & \bar{z} \cdot I_{b_\tau} \end{pmatrix}. \end{aligned}$$

Note that it is crucial that the embedding of E into $\mathbb{C}$ on τ-factor is given by τ.

(4) Prove that $(x, y) \mapsto \{x, h_0(i) y\}$ defines a positive definite symmetric form on $V_{\mathbb{R}}$.

(5) Consider the conjugation action of $U_F(V)(F \otimes_{\mathbb{Q}} \mathbb{R})$ on the set of homomorphisms $\{\mathbb{C}^\times \to \mathrm{GU}(V)(\mathbb{R})\}$. Show that the orbit of h_0 under this action is the product

$$\prod_{\tau \in \Phi} \big(U(a_\tau, b_\tau)/(U(a_\tau) \times U(b_\tau)) \big)$$

Note that $\mathrm{GU}(V)(\mathbb{R})$ also acts on this set, is the $\mathrm{GU}(V)(\mathbb{R})$-orbit of h_0 the same as the $U_F(V)(F \otimes_{\mathbb{Q}} \mathbb{R})$-orbit?

Moving onto the moduli problem of unitary Shimura varieties: let K denote an open compact subgroup of $\mathrm{GU}(V)(\mathbb{A}_f)$. We need to take a large enough subfield $L \subset \mathbb{C}$ that contains all complex embeddings of E. Consider the moduli functor $\mathcal{M}_K : \mathbf{Sch}_{/L}^{\text{loc. noe.}} \to \mathbf{Sets}$ such that for each (connected for simplicity) locally noetherian L-scheme S, $\mathcal{M}_K(S)$ is the equivalence class of tuples (A, i, λ, η) where

- A is an abelian scheme of dimension n over S satisfying a signature condition we specify below,
- $i : \mathcal{O}_E \to \mathrm{End}_S(A)$ is an embedding,
- $\lambda : A \to A^\vee$ is an $\mathcal{O}_E$-linear quasi-isogeny such that $m\lambda$ is a polarization for some $m \in \mathbb{Z}_{>0}$, and such that the Rosati involution induces complex conjugation on $\mathcal{O}_E$, and
- η is a $\pi_1^{\mathrm{et}}(S, \bar{s})$-stable K-orbit of $\mathbb{A}_{E,f}$-linear isomorphisms $V \otimes \mathbb{A}_f \xrightarrow{\simeq} \widehat{V}(A)$ such that there exists $c \in \mathbb{A}_f^\times$ to make the following diagram commute:

$$\begin{array}{ccccc} V \otimes \mathbb{A}_f & \times & V \otimes \mathbb{A}_f & \xrightarrow{\{-,-\}} & \mathbb{A}_f \\ \downarrow \eta & & \downarrow \eta & & \downarrow \times c \\ \widehat{V}(A) & \times & \widehat{V}(A) & \xrightarrow{\lambda\text{-Weil pairing}} & \mathbb{A}_f(1). \end{array}$$

Here we say (A, i, λ, η) is equivalent to $(A', i', \lambda', \eta')$ if there is an $\mathcal{O}_E$-linear quasi-isogeny $\alpha : A \to A'$ (compatible with i and i'), such that $\eta' = \alpha \circ \eta$ and the following diagram of quasi-isogenies on polarization hold

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & A^\vee \\ \downarrow \alpha & & \uparrow \alpha^\vee \\ A' & \xrightarrow{\lambda'} & A'^\vee. \end{array}$$

(6) Show that the λ-Weil pairing also satisfies the condition $\{x, y\} = -\{y, x\}$ and $\{ax, y\} = \{x, \bar{a}y\}$ for $x, y \in \widehat{V}(A)$.

The signature condition can be explained as follows. Consider the exact sequence

$$0 \to \omega_{A^\vee/S} \to H_1^{\mathrm{dR}}(A/S) \to \mathrm{Lie}_{A/S} \to 0 \tag{2.1.1}$$

of $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{O}_S$-modules. As S is over L, we have a decomposition

$$\mathcal{O}_E \otimes \mathcal{O}_S \cong \prod_{\tau : E \to L} \mathcal{O}_S.$$

It follows that (2.1.1) is the direct sum of

$$0 \to \omega_{A^\vee/S,\tau} \to H_1^{\mathrm{dR}}(A/S)_\tau \to \mathrm{Lie}_{A/S,\tau} \to 0 \tag{2.1.2}$$

The polarization λ induces a perfect pairing

$$H_1^{\mathrm{dR}}(A/S) \times H_1^{\mathrm{dR}}(A/S) \to \mathcal{O}_S \quad \text{thus} \quad H_1^{\mathrm{dR}}(A/S)_\tau \times H_1^{\mathrm{dR}}(A/S)_{c\circ\tau} \to \mathcal{O}_S$$

(Note that the Rosati involution condition dictates that the pairing is between τ and $c\circ\tau$-factor.) This induces the duality of (2.1.2) for τ and that for $c\circ\tau$.

(7) Imitate the lecture, give the signature condition for this moduli problem. Observe that in the moduli problem, this is the only place where we need to require that S is defined over L (as opposed to $\mathbb{Q}$). What is the smallest field that this condition can be stated on? For example, we need only to take L to be the Galois closure of E in $\mathbb{C}$, call it E^{Gal} but can we take L even smaller? Show that the smallest field, namely the *reflex field* for this moduli problem is a subfield of E^{Gal} that is fixed by the subgroup $H \subset \mathrm{Gal}(E^{\mathrm{Gal}}/\mathbb{Q})$, where

$$H = \{h \in \mathrm{Gal}(E^{\mathrm{Gal}}/\mathbb{Q}) \mid a_{h\circ\tau} = a_\tau \text{ for every } \tau \in \Phi\}.$$

Important: the reflex field is by definition a subfield of $\mathbb{C}$, i.e. a number field that is equipped with a chosen complex embedding.

When K is sufficiently small, the moduli functor $\mathcal{M}_K$ is represented by a variety of dimension $\sum_{\tau\in\Phi} a_\tau b_\tau$ over the reflex field, which we still denote by $\mathcal{M}_K$.

(7) Imitate the definition of Hilbert modular forms and automorphic forms on $\mathrm{GU}(V)$ when E is an imaginary quadratic field, define automorphic bundles on this moduli space $\mathcal{M}_K$ (possibly after base change to E^{Gal}).

Problem 2.2 (Fake moduli problem for $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$)**.** We explain the moduli problem for $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, using a variant of the moduli problem for $G' := (\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2)^{\det\in\mathbb{G}_m}$. Fix a totally real field F. Choose and fix a set of representative $\{\mathfrak{c}_1, \dots, \mathfrak{c}_{h^+}\}$ of the strict ideal class group of F, i.e. the quotient of fractional ideals of F by principal ideals generated by totally positive elements.

(1) For each abelian variety A over some scheme S equipped with a faithful $\mathcal{O}_F$-action, and for an ideal $\mathfrak{c} \subset \mathcal{O}_F$, the following definition of abelian variety $A \otimes_{\mathcal{O}_F} \mathfrak{c}$ makes sense: choose an element $\delta \in \mathcal{O}_F$ such that $\delta\mathcal{O}_F \subseteq \mathfrak{c}$, so that $\delta\mathfrak{c}^{-1}$ is a genuine ideal of $\mathcal{O}_F$. Let

$$H := A[\delta\mathfrak{c}^{-1}] = \{x \in A \mid \text{for every } a \in \delta\mathfrak{c}^{-1},\ a\cdot x = 0_A\}$$

be the subgroup of A killed by elements in $\delta\mathfrak{c}^{-1}$. We define $A\otimes_{\mathcal{O}_F}\mathfrak{c} := A/H$. Show that this $A \otimes_{\mathcal{O}_F} \mathfrak{c}$ is independent of the choice of δ, and carry a natural action of $\mathcal{O}_F$.

More canonically, we view A as a group functor on all S-schemes: for an S-scheme $A(T) := \mathrm{Hom}_S(T, A)$, then $(A\otimes_{\mathcal{O}_F}\mathfrak{c})(T) := \mathrm{Hom}_S(T, A) \otimes_{\mathcal{O}_F} \mathfrak{c}$ is a group functor represented by an abelian variety (as constructed above).

Let D denote the discriminant of F, and $\mathfrak{d}_F$ the different ideal of F. For each i, $\mathcal{M}_{\mathfrak{c}_i}$ is the moduli space over $\mathbb{Z}[\frac{1}{DN}]$, such that for every $\mathbb{Z}[\frac{1}{DN}]$-scheme S, $\mathcal{M}_{\mathfrak{c}_i}(S)$ is the isomorphism classes of triples (A, λ, i) such that

- A is an abelian scheme over S of dimension $[F:\mathbb{Q}]$, equipped with an action of $\mathcal{O}_F$,
- $\lambda : A\otimes_{\mathcal{O}_F}\mathfrak{c}_i \xrightarrow{\simeq} A^\vee$ is an $\mathcal{O}_F$-equivariant polarization ($\mathfrak{c}_i$-polarization),[1] and
- $i : \mathfrak{d}_F^{-1}\otimes_{\mathbb{Z}}\mu_N \to A[N]$ is an embedding of group scheme over S. (Twisting by $\mathfrak{d}_F^{-1}$ will not affect this definition, but it will benefit our later discussion of compactifications.)

[1]Rigorously speaking, a $\mathfrak{c}_i$-polarization is an isomorphism $\lambda : A\otimes_{\mathcal{O}_F}\mathfrak{c}_i \simeq A^\vee$ such that the natural morphism $\mathfrak{c}_i \to \mathrm{Hom}_{\mathcal{O}_F}(A, A\otimes\mathfrak{c}_i) \xrightarrow{\lambda} \mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)$ induces an isomorphism between $\mathfrak{c}_i$ with "symmetric" elements in $\mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)$ and totally positive elements $\mathfrak{c}_i^+$ in $\mathfrak{c}_i$ with polarizations in $\mathrm{Hom}_{\mathcal{O}_F}(A, A^\vee)$. Here symmetric morphism $\alpha : A \to A^\vee$ means that the dual morphism $A \cong A^{\vee\vee} \xrightarrow{\alpha^\vee} A^\vee$ is the same as α.

Define $\mathcal{M} := \coprod_i \mathcal{M}_{\mathfrak{c}_i}$; it is a smooth scheme over $\mathbb{Z}[\frac{1}{DN}]$ of dimension $[F:\mathbb{Q}]$.

(2) Prove that if $\mathfrak{c}$ and $\mathfrak{c}'$ are two ideals in the same strict ideal class. Show that there is an (not quite canonical) isomorphism $\mathcal{M}_{\mathfrak{c}} \simeq \mathcal{M}_{\mathfrak{c}'}$.

(3) Show that $\mathcal{O}_F^{\times,>0}$ (totally positive units) acts on each $\mathcal{M}_{\mathfrak{c}}$ by sending

$$(A,\lambda,i) \mapsto (A,u\lambda,i) \quad u \in \mathcal{O}_F^{\times,>0}.$$

Let $\mathcal{O}_{F,N}^\times$ denote the subgroup of $\mathcal{O}_F^\times$ consisting of elements that are congruent to 1 modulo N. Show that the action of the subgroup $(\mathcal{O}_{F,N}^\times)^2$ is trivial on each $\mathcal{M}_{\mathfrak{c}}$.

The Shimura variety associated to $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$ with $\Gamma_1(N)$-level structure is isomorphic to

$$Y_1(N) := \mathcal{M}\Big/\Big(\mathcal{O}_F^{\times,>0}\big/\big(\mathcal{O}_{F,N}^\times\big)^2\Big)$$

A reference for more general level structure and for the complex points of this moduli problem is section 2.3 of Yichao Tian and Liang Xiao, *p*-adic cohomology and classicality of overconvergent Hilbert modular forms, in Astérisque 382 (2016), 73–162.

(4) The polarization $\lambda : A \otimes_{\mathcal{O}_F} \mathfrak{c}_i \xrightarrow{\simeq} A^\vee$ induces an $\mathcal{O}_M$-linear perfect pairing

$$H^1_{\mathrm{dR}}(A/\mathcal{M}) \times \big(H^1_{\mathrm{dR}}(A/\mathcal{M}) \otimes_{\mathcal{O}_F} \mathfrak{c}_i^{-1}\big) \to \mathcal{O}_{\mathcal{M}},$$

which in turn defines a natural $\mathcal{O}_M \otimes_{\mathbb{Z}} \mathcal{O}_F$-linear isomorphism

$$\wedge^2_{\mathcal{O}_{\mathcal{M}}\otimes_{\mathbb{Z}}\mathcal{O}_F} H^1_{\mathrm{dR}}(A/\mathcal{M}) \cong \mathcal{O}_{\mathcal{M}} \otimes_{\mathbb{Z}} \mathfrak{c}_i\mathfrak{d}_F^{-1}$$

Explain where the factor $\mathfrak{d}_F$ comes from.

(5) Let L denote the Galois closure of $F(\sqrt{u};\ u \in \mathcal{O}_F^{\times,>0})$ inside $\mathbb{C}$, and let $\mathcal{O}_L$ denote the ring of integers of L. We base change $\mathcal{M}$ to $\mathcal{O}_L$ to define line bundles ω_τ and $\epsilon_\tau := \wedge^2_{\mathcal{O}_{\mathcal{M}}}(\mathcal{H}^1_{\mathrm{dR}}(A)_\tau)$, for embeddings $\tau : F \to L$. Recall that for a paritious weight $\kappa = ((k_\tau)_{\tau\in\Sigma}, w) \in \mathbb{Z}^\Sigma \times \mathbb{Z}$, we can define a line bundle

$$\omega^\kappa := \bigotimes_{\tau\in\Sigma}\Big(\omega_\tau^{k_\tau} \otimes_{\mathcal{O}_{\mathcal{M}}} \epsilon_\tau^{(w-k_\tau)/2}\Big).$$

In a natural way, we let $\mathcal{O}_F^{\times,>0}$ to act on ω_τ and ϵ_τ by, $u \in \mathcal{O}_F^\times$

- sending a section s of ω_τ to $u^{-1/2}\cdot\langle u\rangle^*(s)$, and
- sending a section s of ϵ_τ to $u^{-1}\cdot\langle u\rangle^*(s)$,

where $\langle u\rangle$ is the action of $\mathcal{O}_F^{\times,>0}$ on $\mathcal{M}_{\mathfrak{c}}$ mentioned above.

Show that the induced action of $\mathcal{O}_F^{\times,>0}$ on ω^κ is compatible with the action on $\mathcal{M}$ and hence we may descent ω^κ to $Y_1(N)$ (but not each individual ω_τ and ϵ_τ).

Hints.

Problem 2.1: (3) Of course, one can probably write an inverse to the association $\langle -, - \rangle \mapsto \{-, -\}$. Here is a fancy proof: a bilinear form $V \times W \to F$ is the same as $\mathrm{Hom}(V, \mathrm{Hom}(W, F))$. So in view of this, all Hermitian forms on V are given by $\mathrm{Hom}_E(V, \mathrm{Hom}_E(V, E)^c)$, where the superscript c means to take the conjugate E-structure. On the other hand, all alternating E-Hermitian $\mathbb{Q}$-valued forms are classified by $\mathrm{Hom}_E(V, \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})^c)$. So the bijection reflects that the trace map relates $\mathrm{Hom}_E(V, E)$ with $\mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$. (To see this, one needs to use Tor-Hom adjunction formula, search wikipedia for this...)

(5) The first part is really a question of asking what the centralizer of h_0 is for the action. The second question is slightly tricky. One way to see the subtlety is through the surjective map $\mathbb{G}_m \times \mathrm{U}(V) \to \mathrm{GU}(V)$, which has kernel $\{\pm 1\}$. Taking the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ Galois cohomology of the corresponding exact sequence, we get

$$\mathbb{R}^\times \times \mathrm{U}(V)(\mathbb{R}) \to \mathrm{GU}(V)(\mathbb{R}) \to H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \{\pm 1\})$$

So potentially, the $\mathrm{U}(V)(\mathbb{R})$-orbit could be different from the $\mathrm{GU}(V)(\mathbb{R})$-orbit. The question is whether there is an element of $\mathrm{GU}(V)(\mathbb{R})$ for which the similitude factor is negative. Show that this happens precisely when all $a_\tau = b_\tau$.

(7) The signature condition is, for $\tau \in \Phi$, $\mathrm{Lie}_{A/S,\tau}$ is a locally free sheaf over $\mathcal{O}_S$ of rank a_τ. One can deduce from this condition that

$$\mathrm{rank}\omega_{A^\vee/S,\tau} = b_\tau, \quad \text{and} \quad \mathrm{rank}\omega_{A^\vee/S,c\circ\tau} = a_\tau.$$

Note that this condition is invariant under the action of the group H. So it can be stated over the subfield $(E^{\mathrm{Gal}})^H$.

Problem 2.2: (5) is essentially product formula. The way we define the action on ω_τ and ϵ_τ can be viewed as a way so that $(\mathcal{O}_{F,N}^\times)^2$ acts naturally (when it induces trivial action on $\mathcal{M}_{\mathfrak{c}_i}$.