**Exercise for Talk 3: Background material on algebraic geometry**

We introduce the notion of group schemes, $G$-tosors, and vector bundles associated to a $G$-torsor and a $G$-representation. Then we move on to discuss in details of Gauss–Manin connections on de Rham local systems, and Griffith transversality. After this, we discuss basics of Hodge structure and variation of Hodge structure.

**Problem 3.1** (Group schemes)**.** (1) Let $\mathbb{G}_a$ denote the additive group, that is, as a functor

$$\begin{aligned} \mathbb{G}_a : \mathbf{Ring} &\longrightarrow \mathbf{Sets} \\ R &\longmapsto (R, +) \end{aligned}$$

Show that $\mathbb{G}_a$ is represented by $\operatorname{Spec} \mathbb{Z}[t]$, what is the multiplication map $m : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ on the level of rings, i.e. what is the map $m^* : \mathbb{Z}[t] \to \mathbb{Z}[t_1] \otimes \mathbb{Z}[t_2]$.

(2) Let $\mathbb{G}_m$ denote the multiplicative group. Answer the same question as (1).

(3) Over $\mathbb{F}_p$, there are three types of (commutative) finite flat group scheme of degree $p$:

- $\mathbb{Z}/p\mathbb{Z}$, the constant group scheme;
- $\mu_p$, the kernel of $\mathbb{G}_m \xrightarrow{\text{multiplication by } p} \mathbb{G}_m$;
- $\alpha_p$, kernel of the Artin–Schreier map $\mathbb{G}_a \xrightarrow{x \mapsto x^p - x} \mathbb{G}_a$ (show that this is a homomorphism of group scheme).

Write out the structure ring in each case and the multiplication map in terms of structure rings.

**Problem 3.2** (de Rham cohomology with log structure)**.** We will provide some additional discussion on the de Rham cohomology with logarithmic poles.

The definition of divisors with simple normal crossings varies slightly from authors to authors. Here's the one from the stack project. Let $X$ be a locally noetherian scheme. A *strict normal crossing divisor* on $X$ is an effective Cartier divisor $D \subset X$ such that for every $p \in D$, the local ring $\mathcal{O}_{X,p}$ is regular and there exists a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ such that $D$ is cut out by $x_1, \ldots, x_r$ in $\mathcal{O}_{X,p}$. Let $U := X - D$. In this case, we define the sheaf of differential forms on $X$ with logarithmic pole along $D$ to be the quasi-coherent subsheaf $\Omega^1_X(\log D)$ of $\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_U$ such that

- over $U$, $\Omega^1_X(\log D)|_U \cong \Omega^1_U$,
- at every $p \in D$, in terms of the coordinate above,

$$\Omega^1_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_{X,p} \simeq \Omega^1_{\mathcal{O}_{X,p}} + \sum_{i=1}^{r} \mathcal{O}_{X,p} \frac{dx_i}{x_i}.$$

For example, $X = \mathbb{A}^1_k$ and $D$ is the origin, $\Gamma(X, \Omega^1_X) = k[x]dx$ and $\Gamma(X, \Omega^1_X(\log D)) = k[x]\frac{dx}{x}$.

Remark: A *normal crossing divisor* on $X$ is an effective divisor $D \subset X$ such that for every $p \in D$, there exists an étale morphism $U \to X$ with $p$ in the image and $D \times_X U$ a strict normal crossing divisor on $U$. Applying the above definition on this étale cover, we can define the sheaf with log poles along $D$ by descent.

All schemes considered below will be over a field $k$ of characteristic zero. (Here and later, for a complex of sheaves $C^\bullet$ on $X$, we will write $H^*(X, C^\bullet)$ to mean the hypercohomology of the complex of sheaves.)

(1) First consider the simpliest case: $U = \mathbb{A}^n$, consider its natural embedding into $X = (\mathbb{P}^1)^n$. Then $D := X - U$ is a divisor with strict normal crossings. If $x_1, \ldots, x_n$ are the coordinates of $U$, and $y_i = x_i^{-1}$ denotes the usual coordinate in other affine charts of $X$.

(1a) Describe the sheaf of differentials with logarithmic poles $\Omega^1_{X/k}(\log D)$.

(1b) Let $j : U \to X$ denote the natural inclusion. Then $\Omega^1_{X/k}(\log D)$ is naturally a subsheaf of $j_*\Omega^1_{U/k}$. Show that setting $\Omega^i_{X/k}(\log D) := \wedge^i_{\mathcal{O}_X}\Omega^1_X(\log D)$, then

$$\big[\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k}(\log D) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{X/k}(\log D)\big]$$

is a natural complex of coherent subsheaf of $j_*\Omega^\bullet_{U/k}$ (namely this subsheaf is closed under differential operators). Moreover, this natural inclusion is a quasi-isomorphism (i.e. inducing isomorphisms on the cohomology).

(1c) Suppose that $k = \mathbb{C}$. Deduce that $H^*_{\mathrm{dR}}(U, \Omega^\bullet_U) \cong H^*(X, \Omega^\bullet_{X/k}(\log D))$. (Note that the term on the LHS is not obviously finite dimensional.) When $k = \mathbb{C}$, using the relation between $\Omega^\bullet_{X/\mathbb{C}}(\log D)$ and $\Omega^\bullet_{X/\mathbb{C}}$, deduce that $H^*(X, \Omega^\bullet_{X/\mathbb{C}}(\log D)) \cong H^*(U(\mathbb{C})^{\mathrm{an}}, \mathbb{C})$.

(2) Now we move to the general situation. Let $X$ be a connected smooth variety over $k$ and $D$ a divisor with simple crossings, and $j : U = X - D \hookrightarrow X$. (The general case can be reduced to this case by resolution of singularities.) Use the following fact: for each $x \in D$, there exists an open neighborhood $V$ of $x$ and an *étale* morphism $V \to (\mathbb{P}^1)^n$ such that the pullback of $(\mathbb{P}^1)^n - \mathbb{A}^n$ is $D \cap V$. Prove that there is a natural quasi-isomorphism

$$\Omega^\bullet_{X/k}(\log D) \xrightarrow{\simeq} j_*\Omega^\bullet_{U/k}$$

(3)* Assume $k = \mathbb{C}$. From this, deduce that

$$H^*(U, \Omega^\bullet_U) \cong H^*(X, \Omega^\bullet_{X/\mathbb{C}}(\log D)) \cong H^*(U(\mathbb{C})^{\mathrm{an}}, \mathbb{C}).$$

(4) If $n = \dim X$, show that $\Omega^n_{X/k}(\log D) \cong \Omega^n_{X/k}(D)$. This does not holds for intermediate differential sheaves in general.

This problem is taken from Grothendieck's famous paper: On the de Rham cohomology of algebraic varieties.

**Problem 3.3** (Definition of Gauss–Manin connection)**.** Let $k$ be a field of characteristic zero, and let $X$ be a smooth variety over $k$, and $f : Y \to X$ a proper smooth family. Then recall that we have an exact sequence of differential forms:

$$0 \to f^*\Omega^1_{X/k} \to \Omega^1_{Y/k} \to \Omega^1_{Y/X} \to 0.$$

In particular, the relative differential $\mathcal{O}_Y \xrightarrow{d} \Omega^1_{Y/X}$ factors through $\Omega^1_{Y/k}$.

(1) We first consider the case when $X$ is a smooth curve over $k$. In this case, $f^*\Omega^1_{X/k}$ is locally free of rank 1. We give a filtration on the *full* de Rham complex $\Omega^\bullet_{Y/k}$ by

$$F^0\Omega^i_{Y/k} = \Omega^i_{Y/k} \ \supset \ F^1\Omega^i_{Y/k} = f^*\Omega^1_{X/k} \otimes \Omega^{i-1}_{Y/X} \ \supset \ F^2 = 0.$$

Convince yourself that $f^*\Omega^1_{X/k} \otimes \Omega^{i-1}_{Y/X}$ is a natural subsheaf of $\Omega^i_{Y/k}$ (this is where rank $f^*\Omega^1_{X/k}$ is used). Show that the differential maps on $\Omega^\bullet_{X/k}$ preserves this filtration. Moreover, $F^1\Omega^\bullet_{Y/k}$ is the *same* as $f^*\Omega^1_{X/k} \otimes \Omega^{\bullet-1}_{Y/X}$ (note the shift by one).

Now, we abstractly present this situation as:

(3.3.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^1\Omega^\bullet_{Y/k} & \longrightarrow & \Omega^\bullet_{Y/k} & \longrightarrow & \mathrm{gr}^0\Omega^\bullet_{Y/k} & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & f^*\Omega^1_{X/k} \otimes \Omega^{\bullet-1}_{Y/X} & & & & \Omega^\bullet_{Y/X} & & \end{array}$$

Taking derived pushforward of this sheaf of complexes, show that the connecting map

$$R^n f_*\big(\mathrm{gr}^0\Omega^\bullet_{Y/k}\big) \to R^{n+1} f_*\big(F^1\Omega^\bullet_{Y/k}\big) \tag{3.3.2}$$

defines a map $\nabla_{\mathrm{GM}} : \mathcal{H}^n_{\mathrm{dR}}(Y/X) \to \mathcal{H}^n_{\mathrm{dR}}(Y/X) \otimes \Omega^1_{X/k}$. In addition, show that this is a differential map in the sense that

$$\nabla_{\mathrm{GM}}(a \otimes x) = x da + a\nabla_{\mathrm{GM}}(x)$$

for $a$ a section of $\mathcal{O}_X$ and $x$ a section of $\mathcal{H}^n_{\mathrm{dR}}(Y/X)$. Moreover, show that $\nabla_{\mathrm{GM}}$ satisfies the Griffith transversality condition.

(2)* Now consider the general case when $X$ has dimension, say $r$ over $k$. Provide a natural filtration on the de Rham complex $\Omega^\bullet_{Y/k}$ so that the subquotients of $\Omega^i_{Y/k}$ are (in order)

$$f^*\Omega^r_{X/k} \otimes \Omega^{i-r}_{Y/X}, \quad f^*\Omega^{r-1}_{X/k} \otimes \Omega^{i-r+1}_{Y/X}, \quad \cdots, \quad f^*\Omega^1_{X/k} \otimes \Omega^{i-1}_{Y/X}, \quad \Omega^i_{Y/X}.$$

Using only the extension of the last two terms here, show that we can already define a differential map

$$\nabla_{\mathrm{GM}} : \mathcal{H}^n_{\mathrm{dR}}(Y/X) \to \mathcal{H}^n_{\mathrm{dR}}(Y/X) \otimes \Omega^1_{X/k}.$$

Convince yourself that, via the same reasoning as above, $\nabla_{\mathrm{GM}}$ is a connection, and it satisfies Griffith transversality.

In general, using filtered version of spectral sequence, see that the $n$th row of the $E_1$-page of the spectral sequence is

$$\mathcal{H}^n_{\mathrm{dR}}(Y/X) \xrightarrow{d_1} \mathcal{H}^n_{\mathrm{dR}}(Y/X) \otimes \Omega^1_{X/k} \xrightarrow{d_1} \mathcal{H}^n_{\mathrm{dR}}(Y/X) \otimes \Omega^2_{X/k} \xrightarrow{d_1} \cdots.$$

Show that the latter maps are induced by the first map as connections. Therefore, the Gauss–Manin connection is integrable. (This has another corollary: a coherent sheaf over a characteristic zero smooth variety with an integrable connection is automatically locally free (see the problem below); so it implies immediately that $\mathcal{H}^n_{\mathrm{dR}}(Y/X)$ is locally free.)

**Problem 3.4** (Coherent sheaf with intergrable connection is locally free)**.** Let $X$ be a smooth variety over a field $k$ of characteristic zero. Let $M$ be a coherent sheaf on $X$ with an integrable connection $\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega^1_X$. The goal is to prove that $M$ is locally free as an $\mathcal{O}_X$-module.

To see this, it is enough to work locally in a formal neighborhood of a point $x$, and hence we may practically replace $X$ with $\operatorname{Spec} k[\![x_1, \dots, x_n]\!]$, and then $M_x$ is a finite $k[\![x_1, \dots, x_n]\!]$-module.

(1) Show that $M$ admitting an integrable connection implies that $M_x$ carries *commuting* differential operators $\partial_{x_1}, \dots, \partial_{x_n}$.

(2) Given any $e \in M_x$, show that the following expression

$$\sum_{a_1,\dots,a_n \in \mathbb{Z}_{\geq 0}} \frac{(-x_1)^{a_1} \cdots (-x_n)^{a_n}}{a_1! \cdots a_n!} \partial^{a_1}_{x_1} \cdots \partial^{a_n}_{x_n}(e)$$

is a (or rather unique) horizontal section of $M_x$ (namely killed by all $\partial_i$), with the same reduction as $e$ modulo $(x_1, \dots, x_n)$.

(3) Prove that $M_x$ is a finite free $k[\![x_1, \dots, x_n]\!]$-module. (A maybe a direct way to prove this is to show that taking horizontal lifts of elements in a basis of $M_x/(x_1, \dots, x_n)$ to $M$, there is no relation among these lift.)

**Problem 3.5** (Gauss–Manin connection for elliptic curves)**.** The goal of this problem is to compute explicitly the Gauss–Manin connection on family of elliptic curves. Let $S$ be an affine scheme.

(1) We start with a general elliptic curve $E/S$, and let $\infty$ denote the zero section of the elliptic curve. Set $U := E\backslash\infty$ and $j : U \to S$ the natural inclusion. Show that the following natural morphisms

$$\big[\mathcal{O}_E \to \Omega^1_{E/R}\big] \longrightarrow \big[\mathcal{O}_E(\infty) \to \Omega^1_{E/R}(2\infty)\big] \longrightarrow \big[j_*\mathcal{O}_U \to j_*\Omega^1_{U/R}\big]$$

induce isomorphisms on $\mathbb{H}^1(E, -)$ (namely the hypercohomology of the complex).

(2) Prove that

$$H^1\big(E, \mathcal{O}_E(\infty) \to \Omega^1_{E/R}(2\infty)\big) \cong H^0\big(E, \Omega^1_{E/R}(2\infty)\big).$$

and show that if we write $y^2 = x^3 + ax + b$ for $a, b \in \Gamma(S, \mathcal{O}_S)$, this cohomology has two basis $\frac{dx}{y}$ and $\frac{xdx}{y}$.

Using the last isomorphism of (1), show that $\frac{dx}{y}$ and $\frac{xdx}{y}$ give a basis of the cokernel of $H^0(U, \mathcal{O}_U) \xrightarrow{d} H^0(U, \Omega^1_{U/R})$.

(3) On the affine part $U$ of $E$, show that there exists $A(x), B(x) \in \Gamma(S, \mathcal{O}_S)[x]$ such that

$$A(x)(x^3 + ax + b) + B(x)(3x^2 + a) = 1.$$

(Explicitly, if $\Delta := 4a^3 + 27b^2$, then $A(x) = \frac{-18ax+27b}{\Delta}$ and $B(x) = \frac{6ax^2-9bx+4a^2}{\Delta}$)

Using this, deduce that

$$\frac{dx}{y} = A(x)ydx + 2B(x)dy,$$

as differentials in $\Omega^1_{U/R}$ (but not in $\Omega^1_{U/k}$ when $S = \operatorname{Spec} k[t]$) (It may simplify the notation if we write $P(x) = x^3 + ax + b$.)

(4) Going through the definition of Gauss–Manin connection in Problem 3.3(1) (and use its compatibility with its restriction to $U$) to give a recipe to compute, for a family of elliptic curve $y^2 = x^3 + a(t)x + b(t)$ with $a(t), b(t) \in k[t]$, the Gauss–Manin connection, in terms of $A(x)$ and $B(x)$ above.

Remark: the computation will be very formidable to implement in practice; in the next problem set, this computation will be done in the context of "Tate curve", where some additional input makes the computation a little bit better.

**Hints.**

Problem 3.2: (1) First consider the case when $n = 1$. The general case is simply the "tensor product" of the case of $n = 1$.

(2) The quasi-isomorphism follows readily from the said étale morphism and part (1). once we note that, for an étale morphism $f : V \to (\mathbb{P}^1)^n$.

$$\Omega^1_V \cong f^*\Omega_{(\mathbb{P}^1)^n} \quad \text{and} \quad \Omega^1_V(\log(V \cap D)) \cong f^*\Omega_{(\mathbb{P}^1)^n}(\log((\mathbb{P}^1)^n - \mathbb{A}^n)).$$

(3) To show that $H^*(X, \Omega^\bullet_{X/\mathbb{C}}(\log D)) \cong H^*(U(\mathbb{C})^{\mathrm{an}}, \mathbb{C})$, we need to compare $\Omega_{X/\mathbb{C}}(\log D)$ with $\Omega_{X/\mathbb{C}}$. More precisely, let $\tilde{D}$ denote the normalization of $D$ and $D_2$ denote the singular points of $D$, that is all the codimension 2 intersections of two local irreducible components of $D$. Then

$$\Omega^\bullet_X \to \Omega^\bullet_X(\log D)$$

(To prove this, it is enough to work near a point $x \in D$, and essentially reduces to the local computation as done for the $\mathbb{A}^n$-case.)

Problem 3.3: One reference is Iovita's notes as posted in the wechat group.

(1) To see that $\nabla_{\mathrm{GM}}$ is a connection, we may go back to the chain complex level defining the connecting homomorphism, we see that we may modify the natural lifting $\tilde{x}$ in $\Omega^\bullet_{Y/k}$ into simply $a\tilde{x}$ which would lift $a \otimes x$. The rest is clear.

For the Griffith transversality, we note that the Hodge filtration on $\mathcal{H}^n(X/Y)$ is given by the image of

$$R^n f_*(\Omega^{\geq i}_{X/Y}) \to R^n f_*(\Omega^\bullet_{X/Y}) = \mathcal{H}^n(X/Y)$$

where $\Omega^{\geq i}_{X/Y} := \big[\Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y} \to \cdots\big]$. Note that we are in characteristic zero, so the Hodge spectral sequence degenerate at $E_1$-term. We deduce Griffith transversality by noting that we have a sub exact sequence of (3.3.1):

$$0 \to f^*\Omega^1_{X/k} \otimes \Omega^{\geq i-1}_{Y/X} \to \Omega^{\geq i}_{Y/k} \to \Omega^{\geq i}_{Y/k} \to 0.$$

The connecting homomorphism of this under $Rf_*$ gives the needed Griffith transversality.

Problem 3.4: This is well-known to experts. A spelled-out proof can be found, for example in Proposition 1.2.6 of Kedlaya

`https://arxiv.org/pdf/0811.0190.pdf`

Problem 3.5: We mostly follow section 3.4 of

`http://swc.math.arizona.edu/aws/2007/KedlayaNotes11Mar.pdf`

(1) For the first isomorphism, consider the exact triangle

$$\big[\mathcal{O}_E \to \Omega^1_{E/R}\big] \longrightarrow \big[\mathcal{O}_E(\infty) \to \Omega_{E/R}(2\infty)\big] \longrightarrow \big[R \xrightarrow{\cdot x} R[x]/(x^2)\big] \xrightarrow{+1}$$

Take the long exact sequence of this exact triangle. Note that $H^2\big(E, \mathcal{O}_E(\infty) \to \Omega_{E/R}(2\infty)\big) = 0$.

To see the latter isomorphism, we can in fact show that

$$\big[\mathcal{O}_E(m\infty) \to \Omega^1_{E/R}((m+1)\infty)\big] \to \big[\mathcal{O}_E((m+1)\infty) \to \Omega^1_{E/R}((m+2)\infty)\big]$$

is a quasi-isomorphism for $m \geq 1$.

(3) Since $E/S$ is a family of elliptic curve, the discriminant $4a^2 + 27b^3$ is invertible over $S$.

(4) Consider the following explicit version of (3.3.1) but only over $U$ (in vertical direction)

$$\begin{array}{ccccc} & & \mathcal{O}_U dt & \longrightarrow & \Omega^1_{U/k[t]} \otimes dt \\ & & \downarrow & & \\ \mathcal{O}_U & \longrightarrow & \Omega^1_{U/k} & \longrightarrow & \Omega^1_{U/k[t]} \otimes dt \\ & & \downarrow & & \\ \mathcal{O}_U & \longrightarrow & \Omega^1_{U/k[t]} & & \end{array}$$

Taking $\frac{dx}{y} = A(x)ydx + 2B(x)dy$ a section of $\Omega^1_{U/k[t]}$, and lift it (first) to $\frac{dx}{y}$ in $\Omega^1_{U/k}$ and then taking differential again. So

$$\nabla_{\mathrm{GM}}\big(A(x)ydx + 2B(x)dy\big) = A'_t(x)ydt \wedge dx + A(x)dy \wedge dx + 2B'_t(x)dt \wedge dy + 2B'_x(x)dx \wedge dy.$$

Writing $\omega = \frac{dx}{y}$, we deduce

$$dx \wedge dt = y\omega \wedge dt, \quad dy \wedge dt = \tfrac{1}{2}(3x^2 + a)\omega \wedge dt, \quad dx \wedge dy = \tfrac{1}{2}(a'(t)x + b'(t))\omega \wedge dt.$$

Grouping everything together, we can express $\Delta_{\mathrm{GM}}(\omega)$ as

$$Q(x)\omega \wedge dt$$

for some polynomial $Q(x) \in \Gamma(S, \mathcal{O}_S)[x]$. (They are sums of differentials with even degree poles at $\infty$.) Then we need to compute the image of this expression in the quotient of the top row of the diagram above, that is to modulo the sheaf generated by

$$d\big(ydt\big) = \tfrac{1}{2}(3x^2 + a)\omega \wedge dt, \quad d\big(xydt\big) = \tfrac{1}{2}x(3x^2 + a)\omega \wedge dt + (x^3 + ax + b)\omega \wedge dt, \quad \ldots$$

This would allow us to bring down the degree of $Q(x)$ in $x$ to a linear function and hence a linear combination of $\omega \wedge dt$ and $x\omega \wedge dt$.

Doing the same thing for $x\omega$ gives the formula for Gauss–Manin connection on the other basis element.