Exercise for Talk 4: Geometric properties of modular forms.

We start with Katz' definition of modular forms, and then use it to define Hecke operators. Also, lightly introduce the notion of Tate curves, and see that evaluating Katz modular forms on Tate curves gives the q-expansion. Then we introduce Kodaira–Spencer isomorphism. At the end, we discuss the relation between de Rham cohomology of local systems and space of modular forms.

Problem 4.1 (q-expansion of U_p-operator)**.** Let $N \geq 4$ be an integer, and let p be a prime number that divides N, say $p^r \| N$ for some $r \geq 1$. In this case, we usually write U_p for the Hecke operator at p.

Recall that the modular curve $Y_1(N)$ classifies, for a $\mathbb{Q}$-scheme S, a pair (E, i) where E is an elliptic curve over S, and $i : \mu_{N,S} \to E[N]$ an embedding.

Let f be a Katz modular form of weight k. Then $U_p(f)$ is the Katz modular form, whose evaluation on a test object (E, i, ω) over a $\mathbb{Q}$-algebra R (such that $\operatorname{Spec} R$ is connected) is

$$U_p(f)(E, i, \omega) = p^{k-1} \sum_{\substack{C \subset E[p] \\ C \not\subseteq \mathrm{Im}(i)}} f(E/C, i_C, \omega_C),$$

where the sum is taken over all subgroups of $E[p]$ of order p that is different from the one in $\Im(i)$, i_C is the embedding $\mu_{N,S} \xrightarrow{i} E[N] \to E/C$ (as $C \not\subseteq \mathrm{Im}(i)$, this is an inclusion), and $\omega_C = \check{\pi}^*(\omega)$ with $\check{\pi}$ the map defined by the factorization $\mathrm{mult}_p : E \to E/C \xrightarrow{\check{\pi}} E$.

Give the q-expansion expression of $U_p(f)$ in terms of that of f.

Problem 4.2 (Kodaira–Spencer isomorphism for Hilbert moduli variety)**.** Let F be a totally real field with discriminant D, and let $N \geq 4$ be an integer. Fix a fractional ideal $\mathfrak{c}$ of F. Let $\mathcal{M}_{\mathfrak{c}}(N)$ over $\mathbb{Z}[\frac{1}{ND}]$ denote the moduli stack introduced in Exercise 2, Problem 2.2, over which there is a universal abelian variety $\mathcal{A}$ that carries a faithful $\mathcal{O}_F$-action, a $\mathfrak{c}$-polarization $\lambda : \mathcal{A} \otimes \mathfrak{c} \xrightarrow{\cong} \mathcal{A}^\vee$, and a closed embedding $i : \mu_N \otimes_{\mathbb{Z}} \mathfrak{d}_F^{-1} \hookrightarrow \mathcal{A}$.

Consider the relative de Rham cohomology of $\pi : \mathcal{A} \to \mathcal{M}_{\mathfrak{c}}(N)$:

$$0 \to \omega_{\mathcal{A}/\mathcal{M}_{\mathfrak{c}}(N)} \to \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{M}_{\mathfrak{c}}(N)) \to \mathrm{Lie}_{\mathcal{A}^\vee/\mathcal{M}_{\mathfrak{c}}(N)} \to 0,$$

which carries an $\mathcal{O}_F$-linear Gauss–Manin connection

$$\nabla : \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{M}_{\mathfrak{c}}(N)) \longrightarrow \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{M}_{\mathfrak{c}}(N)) \otimes \Omega^1_{\mathcal{M}_{\mathfrak{c}}(N)}.$$

It is a fact that this ∇ induces an isomorphism

$$\nabla : \mathrm{gr}^1 \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{M}_{\mathfrak{c}}(N)) \xrightarrow{\cong} \mathrm{gr}^0 \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/\mathcal{M}_{\mathfrak{c}}(N)) \otimes \Omega^1_{\mathcal{M}_{\mathfrak{c}}(N)}.$$

(1) Write $\omega_{\mathcal{A}}$ for $\omega_{\mathcal{A}/\mathcal{M}_{\mathfrak{c}}(N)}$. From this, deduce the Kodaira–Spencer map

$$\mathrm{KS} : \omega_{\mathcal{A}} \otimes_{\mathcal{O}_F \otimes \mathcal{O}_{\mathcal{M}_{\mathfrak{c}}(N)}} \omega_{\mathcal{A}^\vee} \longrightarrow \Omega^1_{\mathcal{M}_{\mathfrak{c}}(N)}$$

It turns out that this map is an isomorphism, as we will prove in later lectures.

(2) Let p be a prime number that does not divide DN. Assume in addition that $\mathfrak{c}$ is coprime to p. Let r be an integer that is divisible by all inertia degrees of primes in F above p. Let $\mathbb{Q}_{p^r}$ denote the (unique) unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_{p^r}$, and let $\mathbb{Z}_{p^r}$ denotes its ring of integers. Then

$$\omega_{\mathcal{A},\mathbb{Z}_{p^r}} \cong \bigoplus_{\tau \in \mathrm{Hom}(F, \mathbb{Q}_{p^r})} \omega_{\mathcal{A},\tau}$$

Show that

$$\Omega^1_{\mathcal{M}_{\mathfrak{c}}(N)_{\mathbb{Z}_{p^r}}/\mathbb{Z}_{p^r}} \cong \bigoplus_{\tau\in \mathrm{Hom}(F,\mathbb{Q}_{p^r})} \omega^{\otimes 2}_{\mathcal{A},\tau}.$$

Problem 4.3 (Computation of the theta map)**.** We mostly follow Katz's paper, p-adic properties of modular schemes and modular forms: pp. 69-190 in Modular Functions of One Variable III, Springer Lecture Notes in Mathematics 350 (1973), Appendix A.1.

The goal is to compute the de Rham cohomology of Tate curve, and use it to compute the θ_{k-1}-map $\omega^{2-k} \to \omega^k$ in terms of q-expansions.

Recall from the previous Exercise that for a general elliptic curve E/S with zero section ∞. There are natural isomorphisms

$$\mathbb{H}^1(E, \mathcal{O}_E \to \Omega^1_{E/R}) \cong \mathbb{H}^1\big(E, \mathcal{O}_E(\infty) \to \Omega^1_{E/R}(2\infty)\big) \cong H^0\big(E, \Omega^1_{E/R}(2\infty)\big),$$

and the latter space has a canonical basis $\omega_{\mathrm{can}} = \frac{dX}{Y}$ and $\eta_{\mathrm{can}} = \frac{XdX}{Y}$ if the elliptic curve is written as $Y^2 = 4X^3 + aX + b$.

Now consider the Tate curve written in terms of $Y^2 = 4X^3 - \frac{E_4(q)}{12}X + \frac{E_6(q)}{216}$ over $\mathbb{Z}[\frac{1}{6}]((q))$. We aim to give a relatively easier way to compute the Gauss–Manin connection for Tate curve, or more precisely the connection associated to $\theta = q\frac{d}{dq}$. That is, for $\xi \in H^1_{\mathrm{dR}}(\mathrm{Tate}_q/\mathbb{C}((q)))$, $\nabla(\xi) = \nabla(\theta)(\xi) \otimes \frac{dq}{q}$, for some $\nabla(\theta)(\xi) \in H^1_{\mathrm{dR}}(\mathrm{Tate}_q/\mathbb{C}((q)))$. The Gauss–Manin connection satisfies the following two principles:

- The pairing $H_1(\mathrm{Tate}_q(\mathbb{C})^{\mathrm{an}}, \mathbb{Q}) \times H^1_{\mathrm{dR}}(\mathrm{Tate}_q/\mathbb{C}((q))) \to \mathbb{C}((q))$ is horizontal for the connection, that is, for $\gamma \in H_1(\mathrm{Tate}_q(\mathbb{C})^{\mathrm{an}}, \mathbb{Q})$ and $\xi \in H^1_{\mathrm{dR}}(\mathrm{Tate}_q/\mathbb{C}((q)))$,

$$q\frac{d}{dq}\langle \gamma, \xi\rangle = \langle \nabla(\theta)\gamma, \xi\rangle + \langle \gamma, \nabla(\theta)\xi\rangle.$$

- The two obvious paths, γ_1 from 0 to τ and γ_2 from 0 to 1, as q varies, is horizontal for the Gauss–Manin connection. (i.e. in the above formula, if γ is one of γ_i, $\nabla(\theta)\gamma = 0$.)

From these two principles, we are left to solve equations:

$$\langle \gamma_i, \nabla(\theta)\omega_{\mathrm{can}}\rangle = q\frac{d}{dq}\langle \gamma_i, \omega_{\mathrm{can}}\rangle$$

But $\langle \gamma_i, \omega_{\mathrm{can}}\rangle$ is computable with explicit formulas.

(1) Using this to show that

$$\nabla(\theta)\begin{pmatrix}\omega_{\mathrm{can}}\\ \eta_{\mathrm{can}}\end{pmatrix} = \begin{pmatrix}-\frac{P}{12} & 1\\ \frac{12\theta P - P^2}{144} & \frac{P}{12}\end{pmatrix}\begin{pmatrix}\omega_{\mathrm{can}}\\ \eta_{\mathrm{can}}\end{pmatrix}, \tag{4.3.1}$$

where $P = \frac{3}{\pi^2}\sum_{m\in\mathbb{Z}}\sum'_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2} = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$; here $\sum'$ is to remove the term $(m,n) = (0,0)$. (Remark: in this process, the classical Legendre relation

$$\langle \gamma_1, \eta_{\mathrm{can}}\rangle \cdot \langle \gamma_2, \omega_{\mathrm{can}}\rangle - \langle \gamma_2, \eta_{\mathrm{can}}\rangle \cdot \langle \gamma_1, \omega_{\mathrm{can}}\rangle = 2\pi i$$

may help reducing the computation. Also, one can prove that $\int_{\gamma_2}\omega_{\mathrm{can}} = \frac{\pi i}{6}P$.)

(2) Recall the construction of Kodaira–Spencer map from Gauss–Manin connection. Show that this implies that the computation above shows that the Kodaira–Spencer map induces an isomorphism

$$\omega^{\otimes 2} \cong \Omega^1_{X_1(N)}(\log C) = \Omega^1_{X_1(N)}(C),$$

where C is the cusp, corresponding to $q = 0$, if we extend ω so that ω_{can} is a basis of ω over the $\mathbb{Z}[\frac{1}{6}][\![q]\!]$ (as opposed to just over $\mathbb{Z}[\frac{1}{6}]((q))$.) Here $\Omega^1_{X_1(N)}(\log C)$ means the sheaf of differentials with a possibly log pole (of the form $\frac{dq}{q}$) at the cusp; in dimension 1, this is nothing but twist the usual differential sheaves by $\mathcal{O}_{X_1(N)}(C)$.

(3) Show that $\nabla : \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathrm{Tate}_q/\mathbb{Z}[\frac{1}{6}]((q))) \to \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathrm{Tate}_q/\mathbb{Z}[\frac{1}{6}]((q)))\otimes\mathbb{Z}[\frac{1}{6}]((q))\frac{dq}{q}$ induces a natural map

$$\theta_{k-1} : \omega^{2-k}_{\mathrm{Tate}_q} \longrightarrow \omega^{k-2}_{\mathrm{Tate}_q} \otimes \mathbb{Z}[\frac{1}{6}]((q))\frac{dq}{q}$$

as follows, given s a section of $\omega^{2-k}_{\mathrm{Tate}_q}$, there is a unique lift of s to a section $\tilde{s}$ of $\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}(\mathrm{Tate}_q, \mathbb{Z}[\frac{1}{6}]((q)))$ such that its image under ∇ belongs to the top filtration of the target

$$\omega^{k-2}_{\mathrm{Tate}_q} \otimes \mathbb{Z}[\frac{1}{6}]((q))\frac{dq}{q},$$

namely $\theta_{k-1}(f)$.

Show that, in terms of q-expansions,

$$\theta_{k-1}(f) = \frac{(-1)^k}{(k-2)!}\theta^{k-1}(f) = \frac{(-1)^k}{(k-2)!}(q\frac{d}{dq})^{k-1}(f).$$

Hints.

Problem 4.3: This is all in Katz's paper, p-adic properties of modular schemes and modular forms: pp. 69-190 in Modular Functions of One Variable III, Springer Lecture Notes in Mathematics 350 (1973).

(1) is discussed in A.1.3 of Katz' paper. The computation is a bit long. Also there seems to be a sign error on the lower left term of the matrix in (A.1.3.16). This sign error was carried over to later computation as well. (Also, (A1.2.2) the normalizing factors seem to be wrong too, our normalizing factors in the notes follow wikipedia.)

(2) is essentially because when evaluating the formula (4.3.1) at $q = 0$, the matrix has become $\begin{pmatrix} -\frac{1}{12} & 1 \\ -\frac{1}{144} & \frac{1}{12} \end{pmatrix}$ (which is square-zero, because it is supposed to be nilpotent by a general theorem fo Deligne). The non-triviality of this matrix says that the Gauss–Manin connection has log pole at the cusp.

(3) We will illustrate this here when $k = 3$. It is easier to work this out in general by oneself than to look at a reference, e.g. page 34 of Coleman's paper, A p-Adic Shimura Isomorphism and p-Adic Periods of Modular Forms, Contemporary Mathematics Volume 165, 1994, 21–51. Some of the formulas are from Katz's Antwerp paper §A.1.4.

In this case, $\nabla : \mathcal{H}^1_{\mathrm{dR}} \to \mathcal{H}^1_{\mathrm{dR}} \otimes \Omega^1_{X_1(N)}$ can be decomposed (non-canonically) as $\omega \oplus \omega^{-1} \to \omega^3 \oplus \omega^1$, where the basis of the source and target of this maps are $(\omega_{\mathrm{can}}, \eta_{\mathrm{can}})$ and $(\omega_{\mathrm{can}} \otimes \frac{dq}{q}, \eta_{\mathrm{can}} \otimes \frac{dq}{q})$. What we try to compute is that, starting with $f \cdot \eta_{\mathrm{can}} \in \omega^{-1}$, lift it a unique class $f \cdot \eta_{\mathrm{can}} + g \cdot \omega_{\mathrm{can}} \in \mathcal{H}^1$ so that $\nabla(f \cdot \eta_{\mathrm{can}} + g \cdot \omega_{\mathrm{can}}) \in \omega^3 \subseteq \mathcal{H}^1 \otimes \Omega^1$. Using (4.3.1), we see that

$$\nabla(\theta)(f\eta_{\mathrm{can}} + g\omega_{\mathrm{can}}) \;=\; \big(\frac{P}{12}f + \theta(f) + g\big)\eta_{\mathrm{can}} \;+\; \big(\frac{12\theta P - P^2}{144}f + \theta(g) - \frac{P}{12}g\big)\omega_{\mathrm{can}}.$$

We need to require that $\frac{P}{12}f + \theta(f) + g = 0$, from which we solve for g. Plugging this into the next term, we see, miraculously (or maybe necessarily), the coefficients on ω_{can} is $-\theta^2(f)$.

In general, the following way seems to be working but I am not sure whether it is an optimal or a natural approach.

Observation: $\eta_{\mathrm{can}} - \frac{P}{12}\omega_{\mathrm{can}}$ is horizontal for ∇, that is, $\nabla(\theta)(\eta_{\mathrm{can}} - \frac{P}{12}\omega_{\mathrm{can}}) = 0$. In fact, this comes more naturally as one notices that

$$\langle \gamma_1, \eta_{\mathrm{can}} - \tfrac{P}{12}\omega_{\mathrm{can}} \rangle = 1, \quad \langle \gamma_2, \eta_{\mathrm{can}} - \tfrac{P}{12}\omega_{\mathrm{can}} \rangle = 0$$

or in other words, $\eta_{\mathrm{can}} - \frac{P}{12}\omega_{\mathrm{can}}$ is basis dual to γ_1.

Now, we want to understand $\nabla(\theta) : \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} \to \mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}}$, when we decompose each $\mathrm{Sym}^{k-2}\mathcal{H}^1_{\mathrm{dR}} \simeq \omega^{k-2} \oplus \omega^{k-4} \oplus \cdots \oplus \omega^{2-k}$ with basis elements

$$\omega_{\mathrm{can}}^{\otimes(k-2)}, \quad \omega_{\mathrm{can}}^{\otimes(k-3)} \otimes \eta_{\mathrm{can}}, \quad \cdots, \quad \eta_{\mathrm{can}}^{\otimes(k-2)}$$

suppose that we are given $f\eta^{\otimes k} \in \omega^{2-k}$, we may lift it first to

$$f(\eta_{\mathrm{can}} - \tfrac{P}{12}\omega_{\mathrm{can}})^{\otimes(k-2)}$$

whose image under $\nabla(\theta)$ can be easily computed to be

$$\theta(f)(\eta_{\mathrm{can}} - \tfrac{P}{12}\omega_{\mathrm{can}})^{\otimes(k-2)}$$

Similarly and more generally, one can notice that for any $g \in \mathbb{C}((q))$

$$\begin{aligned} &\nabla(\theta)\big(g\omega_{\mathrm{can}}^{\otimes a} \otimes (\eta_{\mathrm{can}} - \tfrac{P}{12}\omega_{\mathrm{can}})^{\otimes k-2-a}\big) \\ &= \theta(g)\omega_{\mathrm{can}}^{\otimes a} \otimes (\eta_{\mathrm{can}} - \tfrac{P}{12}\omega_{\mathrm{can}})^{\otimes k-2-a} + a \cdot g \cdot \omega_{\mathrm{can}}^{\otimes(a-1)} \otimes (\eta_{\mathrm{can}} - \tfrac{P}{12}\omega_{\mathrm{can}})^{\otimes k-1-a} \end{aligned}$$

It seems a little coincidental that $\nabla(\theta)(\omega_{\mathrm{can}})$ is precisely $\eta_{\mathrm{can}} - \frac{P}{12}\omega_{\mathrm{can}}$. I don't have a natural explanation of this. Applying this to $g = \theta(f), \theta^2(f), \ldots$, together with earlier computation, one can deduce the needed description of θ_{k-1}.

Here is something I don't understand. Supposedly, I should see representation theory of $\mathfrak{sl}_2$ here, but I still can't quite make it out explicitly.