Exercise for Talk 5: Overconvergent modular forms.

Explain the Hasse invariant and supersingular locus. Discuss the rigid analytic fiber of the ordinary locus and supersingular locus. Define Katz p-adic modular forms, and overconvergent modular forms. Define U_p-operators on these forms and quickly mention Coleman's classicality result. If time permits, discuss the Hilbert case.

Problem 5.1 (Counting supersingular elliptic curves)**.** Assume that the prime $p \geq 7$ for simplicity.[1] Recall that supersingular elliptic curves over $\overline{\mathbb{F}}_p$ are in one-to-one correspondence with their j-invariants. Classically, determine the number of j-invariants uses an explicit form of Hasse invariant, but this can be done in a much more abstractly.

The moduli stack of elliptic curve is $X(\mathrm{SL}_2(\mathbb{Z}))$; its coarse moduli space is given by taking j-invariants $j : X(\mathrm{SL}_2(\mathbb{Z})) \to \mathbb{P}^1$.

We make the computation over a cover. Consider the modular curve $X(\Gamma(5))$.[2] It is a Galois cover of $X(\mathrm{SL}_2(\mathbb{Z}))$ with Galois group $\mathrm{SL}_2(\mathbb{F}_5)$[3] (in the sense of function field extension, as there are ramifications at cusps).[4] When compositing with the j-invariant map, $X(\Gamma(5))$ becomes a Galois cover of $\mathbb{P}^1$ with Galois group $\mathrm{PSL}_2(\mathbb{F}_5) \simeq A_5$.

(1) The ramification degree of the cover $X(\Gamma(5)) \to \mathbb{P}^1$ at $\tau = i$ is 2, at $\tau = e^{2\pi i/3}$ is 3, and at $\tau = \infty$ is 5. Check using Riemann-Hurwtiz formula that the genus of $X(\Gamma(5))$ is zero.

(2) Over $X(\Gamma(5))$, the Kodaira–Spencer isomorphism gives an isomorphism $\omega^{\otimes 2} \cong \Omega^1_{X(\Gamma(5))}(\log C)$, where C is the cusps, namely the (reduced subscheme of) the preimage of $\infty \in \mathbb{P}^1$. Show that the degree of ω on $X(\Gamma(5))$ is 5, and compute the number of supersingular points over $X(\Gamma(5))$.

(3) Pretend everything we did above works over $\overline{\mathbb{F}}_p$. Prove the following statements.

- The j-invariant 0 ($\tau = e^{2\pi i/3}$) corresponds to a supersingular curve over $\overline{\mathbb{F}}_p$, if and only if $p \equiv 2 \bmod 3$.
- The j-invariant 1728 ($\tau = i$) corresponds to a supersingular curve over $\overline{\mathbb{F}}_p$, if and only if $p \equiv 3 \bmod 4$.
- The number of supersingular j-invariants that are not 0 or 1728 is $\lfloor \frac{p}{12} \rfloor$.

Problem 5.2 (q-expansion of Hasse invariants)**.** (1) From the expression of Tate curve $\mathrm{Tate}_q \cong \mathbb{C}_p^\times / q^{\mathbb{Z}}$ viewed as rigid analytic elliptic curve, deduce that

$$1 \to \mu_p \to \mathrm{Tate}_q[p] \to \mathbb{Z}/p\mathbb{Z} \to 1.$$

(2) From this, deduce that, viewing Tate curve over $\mathbb{Z}_p((q))$, the natural map

$$\mathrm{Tate}_q \xrightarrow{\text{mult. by } p} \mathrm{Tate}_{q^p}$$

lifts the Frobenius morphism modulo p.

(3) Show that the Hasse invariant h has q-expansion equal to 1.

Problem 5.3 (finite slope U_p-eigenforms are very overconvergent)**.** Let K^p be an open compact subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}}^{(p)})$, and let X denote the modular curve over $\mathbb{Q}_p$, with level structure

[1] One may use similar argument with $\Gamma(3)$ and $\Gamma(4)$ to get the result for prime $p = 5$.

[2] Here we lied a little. The genuine $X(\Gamma(5))$ by definition is over $\mathbb{Q}(\zeta_5)$, but as we consider everything over $\mathbb{C}$, we make base change $X(\Gamma(5)) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, this will split $X(\Gamma(5))$ into 4 connected component. What we use below is one of the component.

[3] This is not the same as S_5, as S_5 has no center, but $\mathrm{SL}_2(\mathbb{F}_5)$ sits in an exact sequence $0 \to \{\pm 1\} \to \mathrm{SL}_2(\mathbb{F}_5) \to A_5 \to 1$.

[4] This is easy to see on the moduli problem or over $\mathbb{C}$.

$K^p\,\mathrm{GL}_2(\mathbb{Z}_p)$. For $r \in (1/p, 1)$, let $X(r)$ denote the rigid analytic subspace of X by removing the supersingular disks of radius $< r$. Recall that, at least when $\varepsilon < \frac{1}{2}$, the U_p-operator factors as

$$S_k^{\dagger,p^\varepsilon}(K^p) \xrightarrow{\text{restriction}} S_k^{\dagger,p^{\epsilon/p}}(K^p) \xrightarrow{U_p} S^{\dagger,p^\varepsilon}(K^p).$$

Show that if f is an overconvergent form in $S_k^{\dagger,r}(K^p)$ for some r possibly very close to 1^-, and if f is an eigenform of the U_p-operator, then f extends naturally to $X^{\mathrm{rig}}((p^{1/2})^+)$

Problem 5.4 (p-adic weight disks)**.** Let p be an odd prime so that $\exp(p\,\cdot) : \mathbb{Z}_p \to (1+p\mathbb{Z}_p)^\times$ is an isomorphism.

For the a (not necessarily commutative in general) pro-finite group G, its Iwasawa algebra over $\mathbb{Z}_p$ is the inverse limit

$$\mathbb{Z}_p[\![G]\!] := \varprojlim_{H \lhd G} \mathbb{Z}_p[G/H],$$

where the limit takes over all open normal subgroups H of G, and $\mathbb{Z}_p[G/H]$ is the group algebra of G/H. (For example, the group algebra $\mathbb{Z}_p[\mathbb{Z}/p\mathbb{Z}] \cong \mathbb{Z}_p[x]/(x^p - 1)$.)

(1) Show that there is a natural isomorphism

$$\begin{array}{ccc} \mathbb{Z}_p[\![T]\!] & \xrightarrow{\cong} & \mathbb{Z}_p[\![\mathbb{Z}_p]\!] \\ T & \longmapsto & [1] - 1 \end{array}$$

where $[1]$ is the group element $1 \in \mathbb{Z}_p$ in the Iwasawa algebra.

Let $\Delta := (\mathbb{Z}/p\mathbb{Z})^\times$. From this deduce that $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \cong \mathbb{Z}_p[\![T]\!][\Delta]$.

(2) Show that there is a natural bijection:

$$\big\{\text{continuous character } \chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times\big\} \xrightarrow{\cong} \big\{\text{continuous homomorphism } \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \to \mathbb{C}_p\big\}.$$

Note that the latter (via the isomorphism $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \cong \mathbb{Z}_p[\![T]\!][\Delta]$) is precisely the points on $p-1$ open unit disks. This is called the *weight disk*. For a character $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, the image of T under the corresponding homomorphism

$$\mathbb{Z}_p[\![T]\!][\Delta] \cong \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \to \mathbb{C}_p$$

is denoted by T_χ; it marks where the character lies on these open disks.

A key feature of p-adic variation of modular forms is that the weight k of modular forms also vary p-adically in the following sense. For a modular form f of weight k, level Np^r (with $p \nmid N$), and nebentypus character

$$\chi = \chi_N \chi_p : (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/p^r\mathbb{Z})^\times \longrightarrow \mathbb{C}_p^\times,$$

we associate its p-adic weight character:

$$\begin{array}{rccc} x^k\chi_p : & \mathbb{Z}_p^\times & \longrightarrow & \mathbb{C}_p^\times \\ & a & \longmapsto & a^k\chi_p(a). \end{array}$$

(3) What is the $T_{x^k\chi_p}$ for this p-adic weight? Draw a few p-adic weights on the weight disks, as k changes, and as χ_p changes (especially its conductor).

As we see, most points on the weight disks are not of the form $x^k\chi_p$, so they are *non-classical* weights. To motivate the construction of overconvergent modular forms of non-classical weights, we discuss the following question.

(4) Let $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ be a continuous character; then χ is "locally analytic". More precisely, show that if $v_p(T_\chi) > 1/p^{m-1}(p-1)$ (where the p-adic valuation is normalized so that $v_p(p) = 1$), then χ extends to a continuous character

$$\begin{aligned} \chi : (\mathbb{Z}_p + p^m\mathcal{O}_{\mathbb{C}_p})^\times = \mathbb{Z}_p^\times \cdot \big(1 + p^m\mathcal{O}_{\mathbb{C}_p}\big)^\times &\longrightarrow \mathbb{C}_p^\times \\ a \cdot x &\longmapsto \chi(a) \cdot \chi(\exp(p^m))^{(\log x)/p^m}. \end{aligned} \tag{5.4.1}$$

(Our bound is not optimal; what is the optimal bound?)

Remark: We quickly explain the construction of overconvergent modular forms of non-classical weights here. Let $N \geq 4$ be an integer relatively prime to p. Let $\mathcal{X}$ denote the compactified modular curve over $\mathbb{Z}_p$. Let $\mathfrak{X}$ denote the corresponding formal scheme over $\mathbb{Z}_p$, i.e. the completion of $\mathcal{X}$ along its special fiber. (Somehow $\mathfrak{X}$ and $\mathcal{X}$ are mostly the same thing, except that $\mathfrak{X}$ does not have a generic fiber by definition.) Let $\mathfrak{X}^{\mathrm{ord}}$ denote the ordinary locus, that is to remove the supersingular points on $\mathfrak{X}$. (The difference here is that: if we remove supersingular points from $\mathcal{X}$, we are just removing points on the special fiber, yet if we remove supersingular points from $\mathfrak{X}^{\mathrm{ord}}$, it has the effect to remove the entire supersingular disks.) Let $\mathcal{E}$ denote the universal "generalized" elliptic over $\mathfrak{X}$, and $\mathfrak{E}$ the formal version.

- Over the entire $\mathfrak{X}$ or $\mathcal{X}$, there is a natural automorphic line bundle $\omega_{\mathcal{E}}$. One may view this line bundle as a $\mathbb{G}_m$-torsor.
- For every point $x \in \mathfrak{X}^{\mathrm{ord}}(\mathcal{O}_{\mathbb{C}_p})$, let $\mathfrak{E}_x$ denote the associated "generalized" elliptic curve over $\mathcal{O}_{\mathbb{C}_p}$. Then the p^∞-torsion points of $\mathfrak{E}_x$ sits in an exact sequence
$$0 \to \mu_{p^\infty} \to \mathfrak{E}_x[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p \to 0,$$
where μ_{p^∞} is the "connected part" of $\mathfrak{E}_x[p^\infty]$. In addition, under the Weil pairing
$$\mathfrak{E}_x[p^\infty] \times \mathfrak{E}_x[p^\infty] \to \mu_{p^\infty},$$
this filtration on $\mathfrak{E}_x[p^\infty]$ is "self-dual". Moreover, we have any isomorphism $\omega_{\mu_{p^\infty}} \cong \omega_{\mathfrak{E}_x}$.
In fact, everything we just discussed exists over the entire ordinary locus:
$$0 \to \mathfrak{E}[p^\infty]^\circ \to \mathfrak{E}[p^\infty] \to \mathfrak{E}[p^\infty]^{\mathrm{et}} \to 0,$$
this filtration is self-dual under the family of Weil pairing
$$\mathfrak{E}[p^\infty] \times \mathfrak{E}[p^\infty] \to \mu_{p^\infty, X},$$
and we have an isomorphism $\omega_{\mathfrak{E}[p^\infty]^\circ} \cong \omega_{\mathfrak{E}}$.
The key here is that the étale part $\mathfrak{E}[p^\infty]^{\mathrm{et}}$ defines a $\mathbb{Z}_p$-local system, and hence a $\mathbb{Z}_p^\times$-torsor over $\mathfrak{X}$, and by duality $\mathfrak{E}[p^\infty]^\circ$ defines the same torsor. Now, the isomorphism $\omega_{\mathfrak{E}[p^\infty]^\circ} \cong \omega_{\mathfrak{E}}$ can be interpreted as the $\mathbb{G}_m$-torsor $\omega_{\mathfrak{E}}$ over the ordinary locus $\mathfrak{X}^{\mathrm{ord}}$ really comes from a torsor for the much smaller group $\mathbb{Z}_p^\times$ (i.e. the transition functions from one affine chart to another can be taken with values in $\mathbb{Z}_p^\times$ as opposed to invertible functions).

One way to interpret what happens above is that over X^{rig}, as we have a $\mathbb{G}_m$-torsor, we may discuss theory of modular forms for any representations of $\mathbb{G}_m$, that is all integer weights, corresponding to sections of $\omega^{\otimes k}$. However, over $\mathfrak{X}^{\mathrm{ord}}$, we are allowed to consider theory of p-adic modular forms associated to any continuous representations of $\mathbb{Z}_p^\times$, which corresponds to a point on the weight disk.

Now, we move to the overconvergent story. Let $\tilde{h}$ denote a lift of Hasse invariant Recall that we have defined the following neighborhoods of the ordinary locus $X^{\mathrm{ord}} = (\mathfrak{X}^{\mathrm{ord}})^{\mathrm{rig}}$, that is removing all open supersingular disks:

$$X^{\mathrm{rig}} \supseteq X^{\mathrm{rig}}(r) \supseteq X^{\mathrm{ord}}$$

where, for $r \in (1/p, 1)$, $X^{\mathrm{rig}}(r)$ is the locus of X^{rig} where $|\tilde{h}(x)| \in [r, 1]$.

An interesting observation of Andreatta–Iovita–Pilloni–Stevens is that as we move from X^{ord} to X^{rig}, the torsor we discussed above, starts from $\mathbb{Z}_p^\times$ expends gradually into $\mathbb{C}_p^\times$. Explicitly, over (a natural formal model of) $X^{\mathrm{rig}}(r)$ with $v \in (p^{-(p-1)/p^n}, 1)$, there is natural $\mathbb{Z}_p^\times + p^{n-1}\mathcal{O}_{\mathbb{C}_p}$-torsor. So for any non-classical character χ satisfying $v_p(T_\chi) > 1/p^{n-2}(p-1)$, the character χ in (5.4.1) defines a natural line bundle ω^χ over (a formal model of) $X^{\mathrm{rig}}(r)$. Its sections over $X^{\mathrm{rig}}(r)$ is the space of overconvergent modular forms of weight χ.

A more rigorous treatment is in F. Andreatta, A. Iovita, and G. Stevens, On overconvergent modular sheaves and modular forms for $\mathrm{GL}_{2/F}$, Israel Journal of Mathematics volume 201, 299–359 (2014). V. Pilloni, Overconvergent modular forms, Ann. Inst. Fourier 63 (2013), no. 1, pp. 219–239.

Hints.

Problem 5.1: (3) For the first two statements, one can use standard facts about elliptic curves over $\overline{\mathbb{F}}_p$ with these special j-invariants. However, one can deduce this "for free" using an integrality argument: indeed, after some computation, one can show that

$$p-1 = 12\cdot\#\{\text{supersingular } j\text{-inv with } j \neq 0, 1728\} + 4\cdot\begin{cases} 0 & j = 0 \text{ is ss.} \\ 1 & \text{o/w} \end{cases} + 6\cdot\begin{cases} 0 & j = 1728 \text{ is ss.} \\ 1 & \text{o/w} \end{cases}$$

But then $p \equiv 1, 5, 7, 11 \bmod 12$. Looking at the congruence relation modulo 12 gives the needed answer.

Problem 5.3: If $U_p(f) = \alpha f$ for some $\alpha \in \mathbb{Q}_p^\times$, then writing f as $\frac{1}{\alpha}U_p(f)$ extends f into the supersingular disk.

Problem 5.4: (3) One observes that as the conductor of χ_p becomes larger, the points representing the p-adic weight moves towards the boundary of the disks.